JORDAN DECOMPOSITION, II. ANALYTIC APPROACH

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Abstract. This part of the paper is devoted to an analytic description of the main construction arising in Jordan decompositions, and to the completeness problems for the root vectors.

For an arbitrary bounded linear operator A, in a Banach space V, that satisfies the Double Cyclicity Condition (see Part 1) we construct a broader locally convex space V_- such that

(i) the initial Banach space V is densely and continuously embedded in V_-,

(ii) the operator and all possible rational functions of the operator are continuously extendable to V_-, and

(iii) all possible root vectors of A belong to V_-.

We discuss algebraic properties equivalent to the completeness of the system of generalized root vectors and obtain the corresponding Jordan decomposition.

Introduction

In the first part of this paper [12], we constructed a natural extension V_- of the initial space V that contains the initial space V as a dense lineal. The operator A and all possible rational functions of A are extended to the larger space by continuity. In this part of the paper we present an analytic approach to our constructions that makes possible a more precise study of the completeness problems for the generalized root vectors (all these vectors belong to the space V_-). Now we can pose the question of whether these root vectors form a complete system. We show that the answer to this question is equivalent to the "generalized semisimplicity" of a special algebra, in which case we say that the algebra is "semi-intricate." For the case in which the root vectors form a complete system (this means that the related algebra is semi-intricate), we obtain an exact analog of the Jordan decomposition formula that involves integration with respect to a "generalized measure."

The entire paper is organized as follows. Section 1 contains necessary preliminaries. In Sections 2 and 3 we describe the first step in the main construction and present a theory of expansion of vectors into "integrals" over generalized eigenvectors with respect to a generalized measure. All results of these sections are actually well known, we only present them in a way suitable for further considerations. Section 4 is devoted to a geometric description of the space containing all possible root vectors. These sections form the first part of the paper, and its results, notions, and notation are extensively used in the present part, which continues our investigations. In Section 5 we study the above construction in analytic terms and introduce some notions needed in the generalized Gelfand transform. Section 6 is devoted to the generalized Gelfand transform and completeness problems for generalized root vectors. In Section 7 we apply the previous considerations to obtain the Jordan decomposition for a general operator satisfying the DCC condition (see Part I of the paper).

Preliminary versions of these results were presented at various conferences since 1983 including Chernogolovka (1983), Voronezh (1985, 1991), Halle (1988), Novgorod (1989), Oberwolfach (1990), Sapporo (1990), Jerusalem (1991), and Beer-Sheva (1992). A draft of this paper [11] has circulated since 1993 as a preprint of the Max-Planck-Institute (Bonn).

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5. The Algebra $J_\infty(A)$

Consider the algebra $R(A)$ and, for each $\lambda \in \mathbb{C} \setminus \operatorname{Spec} A$, introduce the following seminorms on $R(A)$:

$$\|f(A)\|_{0,\lambda} = \|f(A)\|,$$

$$\|f(A)\|_{1,\lambda} = \sup_{B \in R(A)} \frac{\|(f(A) - f(\lambda)\mathbb{I})B\|_{0,\lambda}}{\|(A - \lambda\mathbb{I})B\|_{0,\lambda}},$$

$$\|f(A)\|_{k,\lambda} = \sup_{B \in R(A)} \frac{\|[f(A) - \sum_{i=0}^{k-1} f^{(i)}(\lambda)(i!)^{-1}(A - \lambda\mathbb{I})^i]B\|_{k-1,\lambda}}{\|(A - \lambda\mathbb{I})^k B\|_{k-1,\lambda}}.$$

The seminorm $\|\cdot\|_{k,\lambda}$ vanishes on the k-dimensional subspace $T(k,\lambda)$ and coincides with the previously considered norm when restricted to the ideal $I(k,\lambda)$, that is, for $f(A) \in I(k,\lambda)$ we have

$$\|f(A)\|_{k,\lambda} = \sup_B \frac{\|f(A)B\|_{k-1,\lambda}}{\|(A - \lambda\mathbb{I})^k B\|_{k-1,\lambda}}.$$

Definition 5.1. Denote by $J_\infty(A)$ the completion of $R(A)$ with respect to the family of seminorms $\{\|\cdot\|_{k,\lambda}\}$.

This definition obviously coincides with that in Section 4.

Proposition 5.2. *$\lambda \in \operatorname{Spec}^k A$ if and only if $\|f(A)\|_{k,\lambda} \geq |f^{(k)}(\lambda)|/(k!)$ for any $f \in \operatorname{Rat}(A)$.*

Proof.

$$\lambda \in \operatorname{Spec}^k A \Longleftrightarrow \lambda \in \operatorname{Spec} A(k,\lambda)$$

$$\Longleftrightarrow \forall g \in \operatorname{Rat} A(k,\lambda) = \operatorname{Rat}(A) \qquad \|g(A(k,\lambda)) : \tilde{J}(k,\lambda) \to \tilde{J}(k,\lambda)\| \geq |g(\lambda)|$$

$$\Longleftrightarrow \forall g \in \operatorname{Rat}(A) \qquad \sup_B \frac{\|g(A)(A - \lambda\mathbb{I})^k B\|_{k-1,\lambda}}{\|(A - \lambda\mathbb{I})^k B\|_{k-1,\lambda}} \geq |g(\lambda)|$$

$$\Longleftrightarrow \forall f(A) \in I(k,\lambda) \qquad \|f(A)\|_{k,\lambda} \geq \frac{|f^{(k)}(\lambda)|}{k!}$$

$$\Longleftrightarrow \forall f \in \operatorname{Rat}(A) \qquad \|f(A)\|_{k,\lambda} \geq \frac{|f^{(k)}(\lambda)|}{k!}. \quad \square$$

Proposition 5.3. *$\mu \in \operatorname{Spec} A(k,\lambda)$ if and only if*

$$\|f(A)\|_{k,\lambda} \geq \begin{cases} \left|\dfrac{f(\mu) - \sum_{i=0}^{k-1} f^{(i)}(\lambda)(\mu-\lambda)^i (i!)^{-1}}{(\mu-\lambda)^k}\right|, & \mu \neq \lambda \\ \left|\dfrac{f^{(k)}(\lambda)}{k!}\right|, & \mu = \lambda, \end{cases}$$

for any $f \in \operatorname{Rat}(A)$.

Proof. Let $\mu \neq \lambda$ (otherwise see the previous proposition).

$$\mu \in \operatorname{Spec} A(k,\lambda)$$

$$\iff \forall g \in \operatorname{Rat} A(k,\lambda) = \operatorname{Rat}(A) \qquad \|g(A(k,\lambda)) : J(k,\lambda) \to J(k,\lambda)\| \geq |g(\mu)|$$

$$\iff \forall g \in \operatorname{Rat}(A) \qquad \sup_B \frac{\|g(A)(A-\lambda\mathbb{I})^k B\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^k B\|_{k-1,\lambda}} \geq |g(\mu)|$$

$$\iff \forall f \in I(k,\lambda) \qquad \sup_B \frac{\|f(A)B\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^k B\|_{k-1,\lambda}} = \|f(A)\|_{k,\lambda} \geq \left|\frac{f(\mu)}{(\mu-\lambda)^k}\right|$$

$$\iff \forall f \in \operatorname{Rat}(A) \qquad \|f(A)\|_{k,\lambda} \geq \left|\frac{f(\mu) - \sum_{i=0}^{k-1} f^{(i)}(\lambda)(\mu-\lambda)^i (i!)^{-1}}{(\mu-\lambda)^k}\right|. \quad \square$$

Lemma 5.4. *If $B \in I(k,\lambda)$ and $C \in R(A)$, then $\|BC\|_{k,\lambda} \leq \|B\|_{k,\lambda}\|(A-\lambda\mathbb{I})^k C\|_{k,\lambda}$.*

Proof.

$$\|BC\|_{k,\lambda} = \sup_D \frac{\|BCD\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^k CD\|_{k-1,\lambda}} \cdot \sup_D \frac{\|(A-\lambda\mathbb{I})^k CD\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^k D\|_{k-1,\lambda}} \leq \|B\|_{k,\lambda}\|(A-\lambda\mathbb{I})^k C\|_{k,\lambda}. \quad \square$$

Actually, many seminorms of the form $\|\cdot\|_{k,\lambda}$ are equivalent.

Proposition 5.5. *For $\lambda \notin \operatorname{Spec}^k A$, the norms $\|\cdot\|_{k,\lambda}$ and $\|\cdot\|_{k-1,\lambda}$ are equivalent on $I(k,\lambda)$.*

Proof. Let $f(A) \in I(k,\lambda)$. Without any conditions on λ we have

$$\|f(A)\|_{k,\lambda} = \sup_B \frac{\|f(A)B\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^k B\|_{k-1,\lambda}} \geq \frac{1}{\|(A-\lambda\mathbb{I})^k\|_{k-1,\lambda}}\|f(A)\|_{k-1,\lambda}.$$

Hence, the norm $\|\cdot\|_{k,\lambda}$ is always stronger than the norm $\|\cdot\|_{k-1,\lambda}$ on $I(k,\lambda)$.

Assume that $\lambda \notin \operatorname{Spec}^k A$. Then $\lambda \notin \operatorname{Spec} A(k,\lambda)$, i.e.,

$$\|(A(k,\lambda) - \lambda\mathbb{I})C\|_{k-1,\lambda} \geq c\|C\|_{k-1,\lambda}$$

for $C \in I(k-1,\lambda)$, or

$$\|(A-\lambda\mathbb{I})(A-\lambda\mathbb{I})^{k-1}D\|_{k-1,\lambda} \geq c\|(A-\lambda\mathbb{I})^{k-1}D\|_{k-1,\lambda}$$

for all $D \in R(A)$. Therefore, by Lemma 5.4 we obtain the estimate

$$\begin{aligned}\|f(A)\|_{k,\lambda} &= \sup_D \frac{\|f(A)D\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^k D\|_{k-1,\lambda}} \leq \frac{1}{c}\sup_D \frac{\|f(A)D\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^{k-1} D\|_{k-1,\lambda}} \\ &\leq \frac{1}{c}\sup_D \frac{\|f(A)\|_{k-1,\lambda}\|(A-\lambda\mathbb{I})^{k-1}D\|_{k-1,\lambda}}{\|(A-\lambda\mathbb{I})^{k-1}D\|_{k-1,\lambda}} = \frac{1}{c}\|f(A)\|_{k-1,\lambda}. \quad \square\end{aligned}$$

Thus we need only the seminorms $\|\cdot\|_{k,\lambda}$ with $\lambda \in \operatorname{Spec}^k A$, because the other seminorms do not influence in the topology.

The Space $j_\infty\{M_k\}$.

Consider a decreasing sequence $\{M_k\}_{k=0}^\infty$ of compact subsets of $\mathbb{C}$, $M_0 \supset M_1 \supset M_2 \supset \dots$, such that

$$M_k \setminus (\bigcap_{i=0}^\infty M_i) \subset \operatorname{Isol} M_k,$$

where $\operatorname{Isol} M_k$ stands for the set of isolated points of M_k.

Definition 5.6. Let $j_\infty\{M_k\}$ be the following locally convex space formed by sequences of functions:

$$j_\infty\{M_k\} = \{(f_k)_{k=0}^\infty |\, f_k : M_k \to \mathbb{C},$$

$$\forall \lambda \in M_k, \sup_{\mu \in M_0 \setminus \{\lambda\}} \frac{|f_0(\mu) - \sum_{i=0}^{k-1} f_i(\lambda)(\mu-\lambda)^i (i!)^{-1}|}{|\mu - \lambda|^k} < \infty\}.$$

The above suprema define a fundamental family of seminorms on $j_\infty\{M_k\}$.

The space $j_\infty\{M_k\}$ somewhat resembles the well-known space of Whitney jets (for the case in which all sets M_k coincide) but in fact is much larger. The principal difference is that the relation in the above suprema is not uniform with respect to λ. Nevertheless, the elements of $j_\infty\{M_k\}$ can be regarded, to some extent, as successive derivatives of the same function (see properties (ii) and (iii) below). In all cases, the restrictions of successive derivatives of a function to the related sets M_k provide important examples of elements of $j_\infty\{M_k\}$.

Let us list several simple properties of this space.

(i) *f_0 is continuous on M_0.*

One must verify the continuity at the nonisolated points of M_0 only, but all such points belong to $\bigcap_{k=0}^\infty M_k$, and hence for every $\lambda \in M_0 \setminus \operatorname{Isol} M_0$ $(\subset M_1)$ and for every $\mu \in M_0$ we obtain

$$|f_0(\mu) - f_0(\lambda)| \le C(\lambda)|\mu - \lambda|,$$

which proves assertion (i).

(ii) *For any chosen $\lambda \in \bigcap_{p=0}^\infty M_p$, the function (of μ) given by*

$$\frac{f_0(\mu) - \sum_{i=0}^{k-1} f_i(\lambda)(\mu-\lambda)^i (i!)^{-1}}{(\mu - \lambda)^k}$$

is continuous on M_0, and its limit as $\mu \to \lambda$ is $f_k(\lambda)/(k!)$. Therefore, the functions

$$f_k|_{\bigcap_{p=0}^\infty M_p}, \quad k \ge 1,$$

are completely determined by the function $f_0|_{M_0 \setminus \operatorname{Isol} M_0}$.

We must prove the continuity at λ only (see (i)). Note that

$$\infty > \sup_{\substack{\mu \in M_0 \\ \mu \ne \lambda}} \Big| \frac{f_0(\mu) - \sum_{i=0}^{k} f_i(\lambda)(\mu-\lambda)^i (i!)^{-1}}{(\mu - \lambda)^{k+1}} \Big|$$

$$= \sup_{\substack{\mu \in M_0 \\ \mu \ne \lambda}} \Big| \frac{1}{\mu - \lambda} \Big(\frac{f_0(\mu) - \sum_{i=0}^{k-1} f_i(\lambda)(\mu-\lambda)^i (i!)^{-1}}{(\mu - \lambda)^k} - \frac{f_k(\lambda)}{k!} \Big) \Big|.$$

Hence, as $\mu \to \lambda$, the limit of

$$\frac{f_0(\mu) - \sum_{i=0}^{k-1} f_i(\lambda)(\mu-\lambda)^i (i!)^{-1}}{(\mu-\lambda)^k} - \frac{f_k(\lambda)}{k!}$$

is zero.

(iii) *f_0 is analytic on $\overset{o}{M}_0$ (the interior of M_0), and $f_i(\lambda) = f_0^{(i)}(\lambda)$ for every $\lambda \in \overset{o}{M}_0$ and for all $i = 0, 1, \dots$.*

The proof readily follows from (i) and (ii).

(iv) *The definition of the space $j_\infty\{M_k\}$ imposes no conditions on the functions $f_k|_{\operatorname{Isol} M_k}$ $(k = 0, 1, \dots)$.*

Definition 5.7. Let $\operatorname{Lim} M$ denote the set $\partial(M \setminus \operatorname{Isol} M)$.

Definition 5.8. Consider the following algebra:

$$\mathcal{A}_0 = \{f : \operatorname{Lim} M_0 \to \mathbb{C} \,|\, f_0|_{\operatorname{Lim} M_0} = f \quad \text{for some} \quad (f_k)_{k=0}^\infty \in j_\infty\{M_k\}\}.$$

Define a fundamental system of seminorms on $\mathcal{A}_0$ as follows:
for $k \geq 0$ and $\lambda \in M_0 \setminus \operatorname{Isol} M_0$,

$$|f|_{k,\lambda} = \sup\{\frac{|f(\mu) - \sum_{i=0}^{k-1} f_i(\lambda)(\mu-\lambda)^i (i!)^{-1}|}{|\mu-\lambda|^k} : \mu \in \operatorname{Lim} M_0,\ \mu \neq \lambda\}.$$

Certainly, the element $(f_k)_{k=0}^\infty \in j_\infty\{M_k\}$ is not unique for a given $f \in \mathcal{A}_0$. However, by (ii) and (iii), any choice of this element does not influence in the above seminorms.

Taking account of (i) and (iii), we see that $\mathcal{A}_0$ is a locally convex subalgebra of the algebra $C_H(M_0 \setminus \operatorname{Isol} M_0)$ of all functions continuous on $M_0 \setminus \operatorname{Isol} M_0$ and holomorphic inside M_0 (this is a Banach algebra equipped with the natural norm $\sup\{|f(x)| : x \in \operatorname{Lim} M_0\}$). Any bounded linear functional on $C_H(M_0 \setminus \operatorname{Isol} M_0)$ can be represented by a Borel measure on $\operatorname{Lim} M_0$. The algebra $\mathcal{A}_0$ contains all polynomials, and hence it is dense in the algebra $C_H(M_0 \setminus \operatorname{Isol} M_0)$. The embedding of $\mathcal{A}_0$ in $C_H(M_0 \setminus \operatorname{Isol} M_0)$ is obviously continuous. Therefore, the space of bounded linear functionals on $C_H(M_0 \setminus \operatorname{Isol} M_0)$ is weakly dense in the space of continuous functionals on $\mathcal{A}_0$, and we can regard the functionals on $\mathcal{A}_0$ as generalized measures on $\operatorname{Lim} M_0$.

(v) *The space $j_\infty\{M_k\}$ is naturally isomorphic to $\mathcal{A}_0 \times (\prod_{k=0}^\infty \mathbb{C}^{|\operatorname{Isol} M_k|})$.*

This isomorphism can be described as follows: for any $(f_k)_{k=0}^\infty$ in $j_\infty\{M_k\}$, the restriction $f_0|_{\operatorname{Lim} M_0}$ belongs to $\mathcal{A}_0$ and $(f_k|_{\operatorname{Isol} M_k})_{k=0}^\infty$ belongs to $\prod_{k=0}^\infty \mathbb{C}^{|\operatorname{Isol} M_k|}$.

(vi) *The dual space $(j_\infty\{M_k\})' = j^\infty\{M_k\}$ is naturally isomorphic to*

$$(\mathcal{A}_0)' \oplus (\sum_{k=0}^\infty \mathbb{C}^{|\operatorname{Isol} M_k|}),$$

i.e., for every continuous linear functional $\varphi : j_\infty\{M_k\} \to \mathbb{C}$, *there exist* $d\mu \in (\mathcal{A}_0)'$ (*a "generalized measure on* $\operatorname{Lim} M_0$") *and complex numbers* $c(k,\lambda)$, $k \geq 0$, $\lambda \in \operatorname{Isol} M_k$, *such that*

$$\varphi(f_k) = \int\limits_{\operatorname{Lim} M_0} f_0(\lambda)\, d\mu(\lambda) \;+ \sum_{\substack{k=0,1,\dots \\ \lambda \in \operatorname{Isol} M_k}} f_k(\lambda)\, c(k,\lambda)$$

(*and only finitely many numbers* $c(k,\lambda)$ *are nonzero*).

6. Generalized Gelfand Transform and Completeness

Consider the space $j_\infty(A) = j_\infty\{\operatorname{Spec}^k A\}$ and introduce the mapping

$$R(A) \ni f(A) \mapsto (f^{(k)}|_{\operatorname{Spec}^k A})_{k=0}^\infty \in j_\infty(A).$$

Proposition 5.3 shows that this mapping is continuous, and therefore it can be extended by continuity to the following mapping, the so-called *generalized Gelfand transform*:

$$\wedge : J_\infty(A) \to j_\infty(A).$$

This transform can be also described as follows:

$$(\widehat{B})_k(\lambda) = \varphi_{k,\lambda}(B) \quad \text{for} \quad k = 0, 1, \dots, \quad \lambda \in \operatorname{Spec}^k A.$$

Definition 6.1. The kernel $\operatorname{Ker} \wedge$ is called the *small radical* of $J_\infty(A)$. The operator A and the algebra $J_\infty(A)$ are said to be *semi-intricate* if the small radical of $J_\infty(A)$ is trivial.

The small radical can be described as follows:

$$\begin{aligned} \operatorname{Ker} \wedge = \{ & B : V \to V \mid \text{there exists a net } \{g_\beta\} \subset \operatorname{Rat}(A) \text{ such that} \\ & \forall k \geq 0,\ \forall \lambda \in \operatorname{Spec}^k A, \quad \|g_\beta(A) - B\|_{k,\lambda} \to 0,\ \ g_\beta^{(k)}(\lambda) \to 0\}. \end{aligned}$$

Theorem 6.2. *The system of generalized root vectors*

$$\{\varphi_{k,\lambda} :\ k = 0, 1, \dots,\ \lambda \in \operatorname{Spec}^k A\}$$

is $\sigma(J^\infty(A), J_\infty(A))$*-complete if and only if the operator* A *is semi-intricate.*

This condition means that, for any net $\{g_\beta\} \subset \operatorname{Rat}(A)$ *such that the net* $\{g_\beta(A)\}$ *is fundamental in every seminorm* $\|\cdot\|_{k,\lambda}$ *and* $g_\beta^{(k)}(\lambda) \to 0$ ($k = 0, 1, \dots,\ \lambda \in \operatorname{Spec}^k A$), *we have* $\lim\limits_\beta g_\beta(A) = 0$.

The proof is obvious.

A Refinement of the Completeness.

It was explained in Section 5 (see Definition 5.8 and Assertion (v)) that any element $(f_k)_0^\infty \in j_\infty(A)$ is completely determined by the functions $f_0|_{\operatorname{Lim}\operatorname{Spec} A}$ and $f_k|_{\operatorname{Isol}\operatorname{Spec}^k A}$, $k \geq 0$. Thus, we can consider the following version of the *generalized Gelfand transform*:

$$\wedge : J_\infty(A) \longrightarrow \mathcal{A}_0 \times \Big(\prod_{k=0}^\infty \mathbb{C}^{|\operatorname{Isol}\operatorname{Spec}^k A|}\Big) :$$

$$(\widehat{B})_k(\lambda) = \varphi_{k,\lambda}(B) \quad \text{for} \quad k = 0,\ \lambda \in \operatorname{Lim}\operatorname{Spec} A \quad \text{and for} \quad k \geq 0,\ \lambda \in \operatorname{Isol}\operatorname{Spec}^k A$$

(we denote both transforms by the same symbol $\wedge$ if this leads to no confusion).

Both versions of the generalized Gelfand transform are injective or noninjective simultaneously. This observation proves the following assertion.

Theorem 6.3. *The system of generalized root vectors* $\{\varphi_{k,\lambda} : k = 0, 1, \ldots, \lambda \in \operatorname{Spec}^k A\}$ *is* $\sigma(J^\infty(A), J_\infty(A))$*-complete if and only if the subsystem*

$$\{\varphi_{0,\lambda} : \lambda \in \operatorname{Lim}\operatorname{Spec} A; \quad \varphi_{k,\lambda} : k \geq 0, \quad \lambda \in \operatorname{Isol}\operatorname{Spec}^k A\}$$

is $\sigma(J^\infty(A), J_\infty(A))$*-complete.*

7. Jordan decomposition

Jordan Decomposition in $J^\infty(A)$**.**

Suppose that the system

$$\{\varphi_{0,\lambda} : \lambda \in \operatorname{Lim}\operatorname{Spec} A; \quad \varphi_{k,\lambda} : k \geq 0, \quad \lambda \in \operatorname{Isol}\operatorname{Spec}^k A\}$$

is $\sigma(J^\infty(A), J_\infty(A))$-complete, i.e., the generalized Gelfand transform is injective. Then the dual mapping $\wedge'$ sends $j^\infty(A)$ onto a $\sigma(J^\infty(A), J_\infty(A))$-dense subspace of $J^\infty(A)$.

Therefore, for any $\Phi \in J^\infty(A)$, there exists a net $\{\psi_\beta\} \subset j^\infty(A)$ such that $\wedge'\psi_\beta \overset{\sigma(J^\infty, J_\infty)}{\longrightarrow} \Phi$.

Every functional ψ_β is defined by the generalized measure $d\mu_\beta \in (\mathcal{A}_0)'$ and by a set of numbers $c_\beta(k, \lambda)$, $k \geq 0$, $\lambda \in \operatorname{Isol}\operatorname{Spec}^k A$ (where only finitely many numbers are nonzero),

$$\psi_\beta((f_k)_{k=0}^\infty) = \int\limits_{\operatorname{Lim}\operatorname{Spec} A} f_0(\lambda)\, d\mu_\beta(\lambda) \quad + \sum_{\substack{k \geq 0 \\ \lambda \in \operatorname{Isol}\operatorname{Spec}^k A}} f_k(\lambda) c_\beta(k, \lambda).$$

Thus, we obtain the following formula:

$$\begin{aligned}\Phi(B) &= \lim_\beta (\wedge'\psi_\beta)(B) = \lim_\beta \psi_\beta(\widehat{B}) \\ &= \lim_\beta \Big(\int\limits_{\operatorname{Lim}\operatorname{Spec} A} \varphi_{0,\lambda}(B)\, d\mu_\beta(\lambda) \quad + \sum_{\substack{k \geq 0, \\ \lambda \in \operatorname{Isol}\operatorname{Spec}^k A}} \varphi_{k,\lambda}(B) c_\beta(k, \lambda) \Big).\end{aligned}$$

This formula can be rewritten as the following *Jordan decomposition in* $J^\infty(A)$:

for every $\Phi \in J^\infty(A)$, there exists a net of generalized measures $\{d\mu_\beta\}_\beta \subset (\mathcal{A}_0)'$ and a net of sequences $\{c_\beta(k, \lambda),\ k \geq 0,\ \lambda \in \operatorname{Isol}\operatorname{Spec}^k A\}_\beta$ (for any β, the related sequence contains only finitely many nonzero elements) such that

$$\Phi = \lim_\beta \Big(\int\limits_{\operatorname{Lim}\operatorname{Spec} A} \varphi_{0,\lambda}\ d\mu_\beta(\lambda) \quad + \sum_{\substack{k \geq 0, \\ \lambda \in \operatorname{Isol}\operatorname{Spec}^k A}} \varphi_{k,\lambda}\ c_\beta(k, \lambda) \Big).$$

Certainly, it is desirable to get rid of the limit in the above formula and obtain an assertion of the following type:

for every $\Phi \in J^\infty(A)$, there exist a "generalized measure $d\mu_\Phi$ on $\operatorname{Lim}\operatorname{Spec} A$" and complex numbers $c_\Phi(k, \lambda)$, $k \geq 0$, $\lambda \in \operatorname{Isol}\operatorname{Spec}^k A$, such that

$$\Phi = \int\limits_{\operatorname{Lim}\operatorname{Spec} A} \varphi_{0,\lambda}\ d\mu_\Phi(\lambda) \quad + \sum_{\substack{k \geq 0, \\ \lambda \in \operatorname{Isol}\operatorname{Spec}^k A}} \varphi_{k,\lambda}\ c_\Phi(k, \lambda).$$

Unfortunately, we cannot even guarantee that the separate limits exist. In other words, the subalgebra $\operatorname{Im}\wedge$ can be nondecomposable into the direct product of its projections to $\mathcal{A}_0$ and $\prod_{k=0}^{\infty}\mathbb{C}^{|\operatorname{Isol}\operatorname{Spec}^k A|}$.

As a partial result, one can prove that the limits $\lim_\beta c_\beta(k,\lambda)$ exist for all possible k and λ. To prove this fact, we take any $k \geq 0$ and $\lambda \in \operatorname{Isol}\operatorname{Spec}^k A$ and choose a function f analytic in a neighborhood of $\operatorname{Spec} A$ such that

(i) f vanishes outside of a small disk centered at λ that contains no other point of $\operatorname{Spec} A$, and

(ii) $f(\mu) = (\mu-\lambda)^k/k!$ in a smaller disk centered at λ.

In this case, the operator $f(A)$ is well defined, and one can readily show that $f(A) \in J_\infty(A)$. We obtain

$$\varphi_{p,\mu}(f(A)) = 0 \quad \text{for all} \quad \mu \neq \lambda \quad \text{and all} \quad p$$
$$\varphi_{p,\lambda}(f(A)) = 0 \quad \text{for all} \quad p \neq k, \quad \varphi_{k,\lambda}(f(A)) = 1.$$

Thus, $\Phi(f(A)) = \lim_\beta c_\beta(k,\lambda)$, and hence the limit on the right-hand side exists for any k and λ.

Jordan Decomposition in the Initial Spaces.

Here we mainly repeat the considerations of Section 3 to obtain Jordan decompositions in the initial spaces.

We return to the spaces V and V' and to the inclusions τ_Δ and τ^∇,

$$R(A) \xrightarrow{\tau_\Delta} V, \qquad V' \xleftarrow{\tau^\nabla} R(A).$$

We can readily see that the mappings τ_Δ and τ^∇ are continuous if $R(A)$ is equipped with the system of seminorms $\|\cdot\|_{n,\lambda}$ and the spaces V and V' are equipped with the weak topologies. Thus, we again obtain a rigging

$$J_\infty(A) \xrightarrow{\tau_\Delta} V \xrightarrow{(\tau^\nabla)'} J^\infty(A), \qquad J^\infty(A) \xleftarrow{(\tau_\Delta)'} V' \xleftarrow{\tau^\nabla} J_\infty(A).$$

Let $\mathcal{V}_+$ ($\mathcal{V}^+$) denote the range of τ_Δ (of τ^∇) equipped with the topology transferred from $J_\infty(A)$ by τ_Δ (by τ^∇). We obtain dense inclusions

$$\mathcal{V}_+ \subset V, \qquad V' \supset \mathcal{V}^+.$$

Let $\mathcal{V}^-$ ($\mathcal{V}_-$) denote the space of continuous linear functionals on the space $\mathcal{V}_+$ ($\mathcal{V}^+$). The dualities between $\mathcal{V}_+$ and $\mathcal{V}^-$ and between $\mathcal{V}_-$ and $\mathcal{V}^+$ are extensions of the initial duality between V and V'.

We obtain the following dense inclusions:

$$\mathcal{V}_+ \subset V \subset \mathcal{V}_-, \qquad \mathcal{V}^- \supset V' \supset \mathcal{V}^+.$$

The mapping τ_Δ (τ^∇) is an isomorphism between $J_\infty(A)$ and $\mathcal{V}_+$ ($\mathcal{V}^+$). The mapping $(\tau^\nabla)'$ ($(\tau_\Delta)'$) is an isomorphism between $J^\infty(A)$ and $\mathcal{V}_-$ ($\mathcal{V}^-$).

Set $e_{k,\lambda} = {(\tau^\nabla)'}^{-1}(\varphi_{k,\lambda})$ and $e^{k,\lambda} = (\tau_\Delta)'^{-1}(\varphi_{k,\lambda})$. For every $C \in J_\infty(A)$ we have

$$\langle e_{k,\lambda}, C'\nabla\rangle = \langle {(\tau^\nabla)'}^{-1}(\varphi_{k,\lambda}), \tau^\nabla C\rangle = \varphi_{k,\lambda}(C).$$

Similarly, $\langle C\Delta, e^{k,\lambda}\rangle = \varphi_{k,\lambda}(C)$.

Let us find a version of the Jordan decomposition for vectors in V. This is impossible for all vectors in V, but this turns out to be possible for vectors in a dense lineal $\mathcal{V}_+$ in V.

Since the formulas below are very lengthy, we write LSA instead of $\operatorname{Lim}\operatorname{Spec} A$ and IS^kA instead of $\operatorname{Isol}\operatorname{Spec}^k A$ and omit some subscripts at $d\mu$ and c.

Theorem 7.1. *Let the operator A be semi-intricate. There exist a net of generalized measures $\{d\mu_{\beta;\Delta,\nabla}\}_\beta$ on $\operatorname{Lim}\operatorname{Spec} A$ (the set of continuous linear functionals on the algebra $\mathcal{A}_0$) and a net of sequences of complex numbers $\{c_{\beta;\Delta,\nabla}(k,\lambda),\ k \geq 0,\ \lambda \in \operatorname{Isol}\operatorname{Spec}^k A\}_\beta$ (for any β, only finitely many numbers $c_{\beta;\Delta,\nabla}(k,\lambda)$, $k \geq 0$, $\lambda \in \operatorname{Isol}\operatorname{Spec}^k A$, can be nonzero) such that, for every $x \in \mathcal{V}_+$ and $y \in \mathcal{V}^+$, the following Jordan decomposition holds:*

$$\langle x, y\rangle = \lim_\beta \Big(\int\limits_{LSA} \langle x, e^\lambda\rangle\langle e_\lambda, y\rangle \, d\mu_\beta(\lambda) \ + \sum_{\substack{k=0,1,\dots \\ \lambda\in IS^k A \\ l_1+l_2=k}} \langle x, e^{l_1,\lambda}\rangle\langle e_{l_2,\lambda}, y\rangle c_\beta(k,\lambda)\Big).$$

Similarly, for any $f \in \operatorname{Rat}(A)$, $x \in \mathcal{V}_+$, and $y \in \mathcal{V}^+$ we have

$$\langle f(A)x, y\rangle = \lim_\beta \Big(\int\limits_{LSA} f(\lambda)\langle x, e^\lambda\rangle\langle e_\lambda, y\rangle \, d\mu_\beta(\lambda) + \sum_{\substack{k=0,1,\dots \\ \lambda\in IS^k A \\ l_1+l_2+l=k}} \frac{f^{(l)}(\lambda)}{l!}\langle x, e^{l_1,\lambda}\rangle\langle e_{l_2,\lambda}, y\rangle c_\beta(k,\lambda)\Big).$$

Proof. Consider the linear functional $\Phi_{\Delta,\nabla} : B \mapsto \langle B\Delta, \nabla\rangle$ on $J_\infty(A)$.

It is obvious that $\Phi_{\Delta,\nabla} \in J^\infty(A)$, and therefore it is $\sigma(J^\infty(A), J_\infty(A))$-approximable by $\wedge'$-images of a net $\{d\mu_\beta, c_\beta(k,\lambda)\}_\beta$ of functionals on $j_\infty(A)$,

$$\wedge'\,(d\mu_\beta, c_\beta(k,\lambda)) \overset{\sigma(\mathcal{V}_-,\mathcal{V}^+)}{\longrightarrow} \Phi_{\Delta,\nabla}.$$

Take any $x \in \mathcal{V}_+$ and $y \in \mathcal{V}^+$. Then there are $B, C \in J_\infty(A)$ such that $x = B\Delta$, $y = C'\nabla$, and

$$\begin{aligned}
\langle x, y\rangle \ &= \ \langle B\Delta, C'\nabla\rangle \ = \ \langle (CB)\Delta, \nabla\rangle \ = \varphi_{\Delta,\nabla}(CB) = \lim_\beta\, [\wedge'(d\mu_\beta, c_\beta(k,\lambda)](CB) \\
&= \lim_\beta \Big(\int\limits_{LSA} \varphi_\lambda(CB)\, d\mu_\beta(\lambda) + \sum_{\substack{k=0,1,\dots \\ \lambda\in IS^k A}} \varphi_{k,\lambda}(CB) c_\beta(k,\lambda)\Big) \\
&= \lim_\beta \Big(\int\limits_{LSA} \varphi_\lambda(C)\varphi_\lambda(B)\, d\mu_\beta(\lambda) + \sum_{\substack{k=0,1,\dots \\ \lambda\in IS^k A \\ l_1+l_2=k}} \varphi_{l_2,\lambda}(C)\varphi_{l_1,\lambda}(B) c_\beta(k,\lambda)\Big) \\
&= \lim_\beta \Big(\int\limits_{LSA} \langle B\Delta, e^\lambda\rangle\langle e_\lambda, C'\nabla\rangle\, d\mu_\beta(\lambda) + \sum_{\substack{k=0,1,\dots \\ \lambda\in IS^k A \\ l_1+l_2=k}} \langle e_{l_2,\lambda}, C'\nabla\rangle\langle B\Delta, e^{l_1,\lambda}\rangle c_\beta(k,\lambda)\Big) \\
&= \lim_\beta \Big(\int\limits_{LSA} \langle x, e^\lambda\rangle\langle e_\lambda, y\rangle\, d\mu_\beta(\lambda) + \sum_{\substack{k=0,1,\dots \\ \lambda\in IS^k A \\ l_1+l_2=k}} \langle x, e^{l_1,\lambda}\rangle\langle e_{l_2,\lambda}, y\rangle c_\beta(k,\lambda)\Big)
\end{aligned}$$

The second assertion can be proved in a similar way. □

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