SOME REMARKS ON QUASI-EQUIVALENCE
OF BASES IN FRÉCHET SPACES

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ABSTRACT. We consider some results related to the problem of quasi-equivalence of absolute bases in a Fréchet space. We show that under some conditions on the matrices, transforming one basis into another, these bases are quasi-equivalent.

INTRODUCTION

Let E be a Fréchet space, let $\{] \cdot [_p,\, p = 1, 2, \dots\}$ be a fundamental system of seminorms in E.

Let $(e_i)_1^\infty$ be an **absolute basis** in E. This means that there exists the system $(e^i)_1^\infty$ of functionals on E, biorthogonal to the basis $(e_i)_1^\infty$, and for any $x \in E$

$$x = \sum_{i=1}^\infty e^i(x) e_i$$

and, moreover, the series

$$\sum_{i=1}^\infty |e^i(x)| \,]e_i[_p$$

are convergent for any $p \geq 1$. One can easily show (using the Open Mapping Theorem), that this condition exactly means that the system of seminorms

$$\|x\|_p = \sum_{i=1}^\infty |e^i(x)| \,]e_i[_p, \quad p = 1, 2, \dots$$

is equivalent to the initial system of seminorms $\{] \cdot [_p, p = 1, 2, \dots\}$ on E.

In other words, the decomposition of elements of E with respect to the absolute basis $(e_i)_1^\infty$ defines an isomorphism of the space E onto the **Köthe sequence space**

$$K(\lambda_{ip}) = \{x = (x^i)_1^\infty : \quad \|x\|_p = \sum_{i=1}^\infty |x^i| \, \lambda_{ip} < \infty\}$$

where $\lambda_{ip} =]e_i[_p$.

Let us try to obtain other absolute bases from this one. There are three obvious ways to do this:

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1. **Scaling** of elements of the basis, i.e., considering a new system $(\gamma_i e_i)_1^\infty$, where $\gamma_i \neq 0$. What you get is obviously an absolute basis.

2. **Renumerating** the elements of the basis, i.e., considering a system $(e_{\sigma(i)})_1^\infty$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation (bijective self-mapping) of the set $\mathbb{N}$ of naturals. What you get is obviously an absolute basis.

3. Applying an **automorphism** $T : E \to E$ to elements of the basis, i.e., considering a new system $(Te_i)_1^\infty$. This is also an absolute basis.

It is worth noting that, opposite to the finite dimensional situation, there are plenty of operations of types 1 and 2 that are **not** operations of type 3.

Two absolute bases are called **quasi-equivalent** if one can be transformed to another by a finite number of operations 1 - 3. Because of obvious commutation relations between these operations, we can actually limit ourselves to one operation of each type.

The notion of quasi-equivalence was introduced by M.M. Dragilev [3], and he discovered the first deep result in this area:

Theorem (Dragilev). *Any two bases of the space $A(D)$ of functions holomorphic in the unit disc (endowed with the natural topology of uniform convergence on compact subsets) are quasi-equivalent.*

Actually, he first showed that every basis in this space is absolute, and then established a remarkable property of this space: it has essentially one basis - up to quasi-equivalence. These important and unexpected results attracted a lot of attention. Soon A.S. Dynin and B.S. Mityagin [5] showed that the absoluteness of every basis is in fact true for any nuclear Fréchet space ($A(D)$ is an example of a nuclear space). Then B.S. Mityagin [13] generalized and extended Dragilev's results and methods to more general spaces and, in particular, he has formulated the following

Quasi-equivalence Conjecture:.

Any two bases in a nuclear Fréchet space are quasi-equivalent.

This conjecture (or, better to say, the related problem) was discussed and repeated in several monographs and surveys, see, e.g., [1, 5, 6, 11-14, 16, 17].

There was a lot of activity in this area, especially in the 60s and 70s. Let us mention a deep paper by B.S. Mityagin [14], where he proved the conjecture for a special class of spaces - centers of Hilbert scales. This result was very nontrivial by itself, but, even more importantly, in this paper he introduced a wealth of new ideas into this problem, and, in particular, he discovered that the problem is essentially of a combinatorial nature.

Soon there came a breakthrough - L. Crone and W. Robinson [1] and, independently, V.P. Kondakov [10] proved that the quasi-equivalence conjecture is true for the so called regular spaces (introduced by M.M. Dragilev [4,5]). Very soon P. Djakov [2] found a very simple geometric proof of this result (in this article we give another simple proof of this result). There was a common belief at that time that the conjecture will be proven very soon. To everybody's great surprise, it is still an open question.

I was working on this problem in the 70s, and then returned to it several times in the 80s and 90s, trying to construct counterexamples, based on an approach I proposed in 1974. This approach was described in my Ph. D. thesis (1975),

but was never published. In this article I describe the approach and the related results hoping that they may be useful in future attempts to prove (or disprove) the Quasi-equivalence Conjecture.

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1. ABSOLUTE BASES IN A FRÉCHET SPACE

Let $(e_i)_1^\infty$, $(f_i)_1^\infty$ be two absolute bases in a Fréchet space E. Then we have the following decompositions:

$$f_i = \sum_j \alpha_i^j e_j, \qquad e_i = \sum_j \beta_i^j f_j, \qquad i = 1, 2, ...$$

This simply means that $\alpha_i^j = e^j(f_i)$, $\beta_i^j = f^j(e_i)$. As it was explained above, the bases generate two systems of seminorms on E

$$\|x\|_p = \sum_{i=1}^\infty |e^i(x)| \,]e_i[_p = \sum_i |e^i(x)| \lambda_{ip}, \qquad p = 1, 2, ...$$

$$|x|_p = \sum_{i=1}^\infty |f^i(x)| \,]f_i[_p = \sum_i |f^i(x)| \mu_{ip}, \qquad p = 1, 2, ...$$

each of these systems is equivalent to the initial system $\{] \cdot [_p, p = 1, 2, ...\}$.

One can assume that $] \cdot [_p \le \frac{1}{2}] \cdot [_{p+1}$, then

$$|\cdot|_p \le \frac{1}{2} |\cdot|_{p+1} \quad \text{and} \quad \|\cdot\|_p \le \frac{1}{2} \|\cdot\|_{p+1}$$

This condition and the above mentioned equivalence imply that

$$(1) \qquad \forall p \; \exists p' \; \forall x \in E \quad |x|_p \le \frac{1}{2} \|x\|_{p'} \le \|x\|_{p'}, \quad \|x\|_p \le \frac{1}{2} |x|_{p'} \le |x|_{p'}$$

Hence,

$$\|e_i\|_{(p')'} \ge \sum_j |\beta_i^j| \, |f_j|_{p'} \ge \sum_j |\beta_i^j| \sum_k |\alpha_j^k| \, \|e_k\|_p \; = \sum_k \|e_k\|_p \sum_j |\beta_i^j \alpha_j^k|$$

(we may change the order of summation since all terms are nonnegative.)

Therefore

$$\sum_j |\beta_i^j \alpha_j^k| < \infty$$

and one can easily verify that the matrices $A = (\alpha_i^j)$ and $B = (\beta_i^j)$ are mutually inverse, and one can multiply them according to the usual rules – the related series are absolutely convergent.

Let us recall several simple facts about boundedness properties of operators generated by matrices. Every matrix $\Gamma = (\gamma_i^j)$ generates a linear operator in the

space of sequences, defined at least on the (usually dense) lineal of finitely supported sequences, we will denote this operator by the same letter Γ :

$$(\Gamma(x^i))^j = \sum_i \gamma_i^j x^i$$

Note that this implies the usual agreement

$$(\Gamma\Theta)_i^j = \sum_q \gamma_q^j \theta_i^q$$

(summation over the lower indices of the first factor and the upper indices of the second factor).

We are interested in the following sequence spaces: l_1 and l_∞. Since one can easily find the extreme points of their unit balls, and since they are in a natural duality, it is very easy to compute the norms of the related operators:

$$(2) \qquad \|\Gamma\|_{l_1 \to l_1} = \sup_i \sum_j |\gamma_i^j|$$

$$(3) \qquad \|\Gamma\|_{l_\infty \to l_\infty} = \sup_j \sum_i |\gamma_i^j|$$

$$(4) \qquad \|\Gamma\|_{l_1 \to l_\infty} = \sup_{i,j} |\gamma_i^j|$$

For any matrix $\Gamma = (\gamma_i^j)$ put $\Gamma_+ = (|\gamma_i^j|)$.

It immediately follows from the above formulas, that

$$(5) \quad \|\Gamma\|_{l_1 \to l_1} = \|\Gamma_+\|_{l_1 \to l_1}, \quad \|\Gamma\|_{l_\infty \to l_\infty} = \|\Gamma_+\|_{l_\infty \to l_\infty}, \quad \|\Gamma\|_{l_1 \to l_\infty} = \|\Gamma_+\|_{l_1 \to l_\infty}$$

As usually, δ_i^j will denote the entries of the identity matrix:

$$\delta_i^j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

The following simple result will be useful in our considerations:

Lemma 1. *Let* $A = (\alpha_i^j), B = (\beta_i^j)$ *be two matrices. A diagonal matrix* $M = (\delta_i^j \mu_i)$ *such that* $\|MA\|_{l_1 \to l_1} \leq \alpha$ *and* $\|BM^{-1}\|_{l_1 \to l_1} \leq \beta$ *exists if and only if* $\|B_+ A_+\|_{l_1 \to l_1} \leq \alpha\beta$.

Proof.

The "only if" part is obvious. Let us prove the "if" part. Assuming that $\|B_+ A_+\|_{l_1 \to l_1} < \infty$, we see that

$$\forall i \qquad \sum_k |\alpha_i^k| \sum_j |\beta_k^j| = \sum_k \sum_j |\beta_k^j \alpha_i^k| = \sum_j \sum_k |\beta_k^j \alpha_i^k| < \infty$$

therefore
$$\forall k \qquad \sum_j |\beta_k^j| < \infty.$$

We choose $\mu_i = \beta^{-1}(\sum_j |\beta_i^j|)$. Then

$$\|BM^{-1}\|_{l_1 \to l_1} = \sup_i \sum_j |\mu_i^{-1}\beta_i^j| = \beta$$

and

$$\|MA\|_{l_1 \to l_1} = \sup_i \sum_j |\alpha_i^j \mu_j| = \sup_i \sum_j |\alpha_i^j|\beta^{-1}\sum_k |\beta_j^k|$$

$$= \beta^{-1}\sup_i \sum_k \sum_j |\alpha_i^j \beta_j^k| = \beta^{-1}\|B_+ A_+\|_{l_1 \to l_1} \le \alpha \qquad \blacksquare$$

Consider the **spectral radius** $\rho(A)$ of a $m \times m$ matrix A,

$$\rho(A) = \{\max |\lambda| : \lambda \in Spec\, A\} = \lim_{n \to \infty} \|A^n\|_{l_1 \to l_1}^{1/n}$$

If A has non-negative entries and if P is a canonical projector on a coordinate subspace, then the entries of PAP do not exceed the related entries of A, therefore $\rho(PAP) \le \rho(A)$ – this immediately follows from the second formula for the spectral radius.

Lemma 2. *Let A be an $n \times n$ matrix.*

(6)
$$\inf_\Lambda \|\Lambda A \Lambda^{-1}\|_{l_1 \to l_1} = \rho(A_+)$$

where Λ is a diagonal $n \times n$ matrix with non-negative entries.

Proof.
   Obviously,
$$Spec\, A_+ = Spec\, \Lambda A_+ \Lambda^{-1}$$

and
$$Spec\, A_+ \subset \{z \in \mathbb{C} : |z| \le \|\Lambda A_+ \Lambda^{-1}\|_{l_1 \to l_1} = \|\Lambda A \Lambda^{-1}\|_{l_1 \to l_1}\}$$

so
$$\rho(A_+) \le \inf_\Lambda \|\Lambda A \Lambda^{-1}\|_{l_1 \to l_1}.$$

Since the transposed matrix A_+^* is non-negative then, by the Frobenius-Perron Theorem, there exists a non-negative eigenvector λ, whose eigenvalue is $\rho(A_+^*) = \rho(A_+)$. Let Λ be the diagonal matrix with this vector λ on the diagonal. If λ has no zero components, then the matrix Λ is invertible, and one can easily verify that

$$\|\Lambda A \Lambda^{-1}\|_{l_1 \to l_1} = \|\Lambda A_+ \Lambda^{-1}\|_{l_1 \to l_1} = \rho(A_+),$$

and the result is proven.

If λ has zero components, we consider the canonical projector P onto the coordinate subspace spanned by the zero coordinates of λ and consider the non-negative matrix PA_+P. As it was explained above,

$$\rho(PA_+P) \leq \rho(A_+)$$

Let λ_1 be the Frobenius-Perron vector for PA_+^*P. If the only zero components of λ_1 are the obvious ones, we consider a diagonal matrix Λ with the diagonal $\lambda + \epsilon\lambda_1$ with a positive ϵ. This matrix is invertible, and one can easily see that

$$\|\Lambda A \Lambda^{-1}\|_{l_1 \to l_1} = \|\Lambda A_+ \Lambda^{-1}\|_{l_1 \to l_1} \to \rho(A_+), \text{ as } \epsilon \to 0^+.$$

So the Lemma is proven for this situation.

If λ_1 has nontrivial zero components, we repeat the same trick, and so on. As a result we are able to prove the Lemma in all situations. ∎

Lemma 3 (Interpolation Lemma). *Let* Λ_p, M_p, $p = 1, 2, ...,$ *be diagonal matrices with nonnegative diagonal elements* λ_{ip}, μ_{ip}, *respectively. Assume that*

$$\forall i, j \qquad 1 \leq \sum_p \frac{\mu_{ip}}{\lambda_{jp}}$$

Then for every matrix A

$$\|A\|_{l_1 \to l_1} \leq \sum_p \|\Lambda_p^{-1} A M_p\|_{l_1 \to l_1}$$

Proof.

$$\|A\|_{l_1 \to l_1} = \sup_i \sum_j |\alpha_i^j|$$

$$\leq \sup_i \sum_j \sum_p \frac{\mu_{ip}}{\lambda_{jp}} |\alpha_i^j| \leq \sum_p \sup_i \sum_j \frac{\mu_{ip}}{\lambda_{jp}} |\alpha_i^j|$$

$$= \sum_p \|\Lambda_p^{-1} A M_p\|_{l_1 \to l_1} \qquad ∎$$

Theorem 1. *The vectors* $f_i = \sum \alpha_i^j e_j$, $i = 1, 2, ...,$ *form an absolute basis in* E *if and only if the matrix* A *is invertible and*

(7) $$\forall p \quad \exists m(p) \quad \|\Lambda_p A_+ (A^{-1})_+ \Lambda_{m(p)}^{-1}\|_{l_1 \to l_1} \leq 1$$

where $A = (\alpha_i^j)_{i,j=1}^\infty$, A^{-1} *is its inverse (all related series are absolutely convergent),* Λ_p *is the diagonal matrix* $(\delta_i^j \lambda_{ip})_{i,j=1}^\infty$

Proof.

We already know that if $\{f_i, i = 1, 2, \dots\}$ form an absolute basis then the matrix A is invertible and all related series are absolutely convergent. Let

$$A = (\alpha_i^j), \ A^{-1} = (\beta_i^j)$$

By Lemma 1, the condition (7) is equivalent to the fact that for every p there exists a diagonal matrix $M_p = (\mu_{jp}\delta_j^i)_{i,j=1}^\infty$, such that

$$(8) \qquad \|\Lambda_p \, A \, M_p^{-1}\|_{l_1 \to l_1} \leq 1$$

and

$$(9) \qquad \|M_p A^{-1} \Lambda_{m(p)}^{-1}\|_{l_1 \to l_1} \leq 1$$

Then (8) simply means that

$$1 \geq \sup_i \sum_j \mu_{ip}^{-1} |\alpha_i^j| \lambda_{jp}$$

or

$$(10) \qquad \forall i \qquad \mu_{ip} \geq \sum_j |\alpha_i^j| \lambda_{jp} = \sum_j |e^j(f_i)| \,]e_j[_p = \|f_i\|_p$$

As for (9), it means that

$$\forall x \in l_1 \qquad \|M_p A^{-1} \Lambda_{m(p)}^{-1} x\|_{l_1} \leq \|x\|_{l_1}$$

or, putting $y = \Lambda_{m(p)}^{-1} x$

$$(11) \qquad \forall y, \ \Lambda_{m(p)} y \in l_1 \qquad \sum_i |\sum_j y^j \beta_j^i \mu_{ip}| \leq \sum_i |y^i| \lambda_{i,m(p)}$$

To complete the proof we must verify the following

Claim. *Conditions (8) - (9) are equivalent to the fact that the vectors $f_i = \sum \alpha_i^j e_j$ form an absolute basis in E.*

Proof of the claim.

Let us first show that conditions (8), (9) imply that $\{f_i, \ i = 1, 2, \dots\}$ is an absolute basis in E.

Take any $y \in E$, decompose it as $y = \sum_i y^i e_i$. Let $\hat{y}$ denote the coefficient sequence $(y^i)_1^\infty$. The series

$$\sum_i |y^i| \lambda_{i,m(p)} = \|y\|_{m(p)} = \|\Lambda_{m(p)} \hat{y}\|_{l_1}$$

converge for every $p \geq 1$, since the basis (e_i) is assumed to be absolute.

Consider the series $\sum_j (\sum_i y^i \beta_i^j) f_j$. The inequality (11) means that for every $y \in E$, $y = \sum_i y^i e_i$ the expression $\sum_j \mu_{jp} |\sum_i y^i \beta_i^j|$ is finite, so the series $\sum_i y^i \beta_i^j$ are convergent.

We are going to verify that

$$y = \sum_j (\sum_i y^i \beta_i^j) f_j$$

and that this series is absolutely convergent in E, i.e., the series

$$\sum_j \|f_j\|_p |\sum_i y^i \beta_i^j|$$

converges for every $p \geq 1$. (Recall that the system of seminorms $\{\|\cdot\|_p,\ p = 1, 2, \dots\}$ is equivalent to the initial one.)

Applying (10) and (11), we get

$$\sum_j \|f_j\|_p |\sum_i y^i \beta_i^j| \leq \sum_j \mu_{jp} |\sum_i y^i \beta_i^j| \leq \|y\|_{m(p)} < \infty$$

Therefore

$$\|y - \sum_{j \leq N} (\sum_i y^i \beta_i^j) f_j\|_p = \|y - \sum_i y^i \sum_{j \leq N} \beta_i^j f_j\|_p$$

$$= \|y - \sum_i y^i (e_i - \sum_{j > N} \beta_i^j f_j)\|_p = \|\sum_i y^i \sum_{j > N} \beta_i^j f_j\|_p$$

$$= \|\sum_{j > N} (\sum_i y^i \beta_i^j) f_j\|_p \leq \sum_{j > N} \|f_j\|_p |\sum_i y^i \beta_i^j|$$

$$\leq \sum_{j > N} \mu_{jp} |\sum_i y^i \beta_i^j| \to 0$$

To prove that the decomposition of $x \in E$ with respect to the system $\{f_i, i = 1, 2, \dots\}$ is unique, we assume the opposite and obtain a decomposition $0 = \sum_i x^i f_i$, where the series is convergent in E. Applying the continuous functionals e^j (whose existence follows from the assumption that $\{e_j, j = 1, 2, \dots\}$ is a basis) to the both sides of the decomposition, we get:

$$0 = \sum_i x^i \alpha_i^j, \ \ j = 1, 2, \dots$$

Since the matrix A is invertible, we get

$$x^i = 0, \ \ i = 1, 2, \dots$$

Now let us show that the fact that $\{f_i, i = 1, 2, \dots\}$ is an absolute basis in E implies (8), (9).

We may take $]\cdot[_p = \|\cdot\|_p$ since the system of seminorms $(\|\cdot\|_p)$ is equivalent to the system $(]\cdot[_p)$. By (1),

$$\forall p \ \ \exists m(p): \ \forall i \ \|f_i\|_p \leq |f_i|_{m(p)}, \ \ \ |e_i|_p \leq \|e_i\|_{m(p)}$$

or

$$\sum_i |\alpha_i^j| \lambda_{ip} \leq \|f_i\|_{m(p)} = \mu_{i, m(p)}$$

$$\sum_j |\beta_i^j| \|f_j\|_p \leq \lambda_{i, m(p)}$$

or

$$\|\Lambda_p A M_{m(p)}^{-1}\|_{l_1 \to l_1} \leq 1, \ \ \ \ \|M_p A^{-1} \Lambda_{m(p)}^{-1}\|_{l_1 \to l_1} \leq 1$$

This completes the proof. ∎

The following theorem is an immediate corollary of the above result:

Theorem 2. *Let $K(\lambda_{ip})$ and $K(\mu_{ip})$ be two Köthe spaces. They are isomorphic to each other if and only if there exist two mutually inverse matrices A and A^{-1} such that for any p there exists $m(p)$, satisfying the following conditions*

$$\|\Lambda_p A^{-1} M_{m(p)}^{-1}\|_{l_1 \to l_1} \leq 1$$

$$\|M_p A \Lambda_{m(p)}^{-1}\|_{l_1 \to l_1} \leq 1$$

where

$$\Lambda_p = (\lambda_{ip}\delta_i^j)_{i,j=1}^\infty, \qquad M_p = (\mu_{ip}\delta_i^j)_{i,j=1}^\infty$$

Nuclear Fréchet spaces, introduced by A. Grothendieck in [8, 9], occupy a very special place among all Fréchet spaces. They have many remarkable properties, giving a foundation to a rather widespread hope that it is possible to develop a reasonable structure theory for, at least, large subclasses of such spaces.

An especially important part of such future theory should be a manageable criterion for two spaces to be isomorphic. Since there are numerous examples of nuclear Fréchet spaces without bases (the first ones constructed by B. Mityagin and myself in 1974, see [15]), a seemingly much more approachable class is formed by nuclear Fréchet spaces with bases. By the Dynin-Mityagin theorem [7], every basis in a nuclear Fréchet space is absolute, so such a space is naturally isomorphic to a Köthe sequence space. Therefore it is very important to obtain a working criterion of isomorphism for Köthe nuclear spaces. Theorem 2 above does not give a good criterion since it reduces the problem of isomorphism to the existence of a pair of mutually inverse matrices, which is only a slight reformulation of the definition of isomorphism. If the Quasi-equivalence Conjecture is true then it does provide, maybe, the best possible criterion of this type, reducing the problem of isomorphism to a purely combinatorial question of existence of a permutation of naturals with the required properties.

The nuclearity of a Köthe space

$$K(\lambda_{ip}) = \{x = (x^i): \quad \sum_i |x^i|\, \lambda_{ip} = |x|_p \ < \infty\}$$

can be expressed as follows (see, e.g., [12]):

$$\forall p \quad \exists p' \quad \sum_i \frac{\lambda_{ip}}{\lambda_{i,p'}} \leq 1$$

We may always assume that $p' = p + 1$, so

$$(12) \qquad \sum_i \frac{\lambda_{ip}}{\lambda_{i,(p+1)}} \leq 1$$

This can be rewritten as follows:

$$\|\Lambda_{p+1}^{-1}\Lambda_p\|_{l_\infty \to l_1} \leq 1$$

Therefore the system of seminorms

$$^\infty|x|_p = \sup_i |x_i|\, \lambda_{ip}$$

is equivalent to the initial one. This immediately implies the following result:

Theorem 2′. *Let $K(\lambda_{ip})$ and $K(\mu_{ip})$ be two* **nuclear** *Köthe spaces. They are isomorphic if and only if there exist two mutually inverse infinite matrices A and A^{-1} such that for any p there exists $m(p)$, satisfying the following conditions*

$$\|\Lambda_p\, A^{-1}\, M_{m(p)}^{-1}\|_{l_1 \to l_\infty} \leq 1$$

$$\|M_p\, A\, \Lambda_{m(p)}^{-1}\|_{l_1 \to l_\infty} \leq 1$$

where

$$\Lambda_p = (\lambda_{ip}\delta_i^j)_{i,j=1}^\infty, \qquad M_p = (\mu_{ip}\delta_i^j)_{i,j=1}^\infty$$

In other words, if $A = (\alpha_i^j)$, $A^{-1} = (\beta_i^j)$, then

$$|\beta_i^j| \leq \inf_p \frac{\mu_{i,m(p)}}{\lambda_{jp}}, \qquad |\alpha_i^j| \leq \inf_p \frac{\lambda_{i,m(p)}}{\mu_{jp}}$$

Problem. *Consider two matrices $(P_i^j), (Q_i^j)$ with non-negative entries. Under what conditions on P, Q does there exist a pair of mutually inverse matrices $(\alpha_i^j), (\beta_i^j)$ dominated by the given ones, i.e., such that*

$$\forall i, j \qquad |\alpha_i^j| \leq P_i^j, \quad |\beta_i^j| \leq Q_i^j \quad ?$$

2. Quasi-equivalent absolute bases in a Fréchet space.

If two absolute bases $(e_i)_1^\infty$ and $(f_i)_1^\infty$ are quasi-equivalent then there exists a sequence of nonzero scalars $(\gamma_i)_{i=1}^\infty$, a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, such that the linear operator T uniquely defined by the conditions

$$Te_i = \gamma_i f_{\sigma(i)}, \ i = 1, 2, \dots$$

is an automorphism of the space E. Note that the inverse operator is uniquely defined by the conditions

$$T^{-1}f_i = \gamma_{\sigma^{-1}(i)}^{-1} e_{\sigma^{-1}(i)}, \quad i = 1, 2, \dots$$

Taking (1) into account we see that this means the following: for every p there exists $n(p)$ such that

$$\forall x \in E \qquad]Tx[_p \leq]x[_{n(p)}, \qquad]T^{-1}x[_p \leq]x[_{n(p)}$$

Since each of the systems of seminorms $(\|\cdot\|_p)_{p\geq 1}$, $(|\cdot|_p)_{p\geq 1}$, is equivalent to the initial system $(]\cdot[_p)_{p\geq 1}$ we may arbitrarily replace seminorms of one system by seminorms of another. So, we may rewrite the above inequalities as follows: for every p there exists $m(p)$ such that

$$\forall x \in E \qquad |Tx|_p \leq \|x\|_{m(p)}, \qquad \|T^{-1}x\|_p \leq |x|_{m(p)}$$

Decomposing $x = \sum_i x^i e_i$ and $x = \sum_i y^i f_i$, we get:

$$\forall x \in E \qquad \sum_i |x^i \gamma_i| \mu_{\sigma(i),p} \leq \sum_i |x^i| \lambda_{i,m(p)}$$

$$\forall x \in E \qquad \sum_i |y^i \gamma^{-1}_{\sigma^{-1}(i)}| \lambda_{\sigma^{-1}(i),p} \leq \sum_i |y^i| \mu_{i,m(p)}$$

This is obviously equivalent to the conditions:

$$\forall i \qquad |\gamma_i| \mu_{\sigma(i),p} \leq \lambda_{i,m(p)}$$

$$\forall i \qquad |\gamma_i^{-1}| \lambda_{ip} \leq \mu_{\sigma(i),m(p)}$$

Eliminating γ_i, we get:

$$\forall i \qquad \frac{\lambda_{ip}}{\mu_{\sigma(i),m(p)}} \leq \frac{\lambda_{i,m(q)}}{\mu_{\sigma(i),q}}$$

So, the quasi-equivalence of the bases under consideration is equivalent to the existence of a function $m : \mathbb{N} \to \mathbb{N}$ and a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that

(13) $$\qquad \forall i \qquad \forall p \qquad \frac{\lambda_{ip} \mu_{\sigma(i),q}}{\lambda_{i,m(q)} \mu_{\sigma(i),m(p)}} \leq 1$$

As it was first observed by B. Mityagin [14], it is sufficient (and certainly necessary) to show the existence of an **injective** mapping $\sigma : \mathbb{N} \to \mathbb{N}$. Then a version of the usual Cantor-Bernstein argument (which proves that two sets can be bijectively mapped one onto another provided each of them can be injectively imbedded into another) gives the existence of the needed bijection (see [14] for details). Let us consider the following set:

$$K^i(p,q;P,Q) = \{n \in \mathbb{N} : \quad \frac{\lambda_{ip} \mu_{nq}}{\mu_{nP} \lambda_{iQ}} \leq 1\}$$

We need to show that there exists a function $m : \mathbb{N} \to \mathbb{N}$ such that there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\sigma(i) \in K^i_m \stackrel{def}{=} \bigcap_{p,q} K^i(p,q;m(p),m(q))$$

It is obvious that such an injection exists only if for every finite subset S of $\mathbb{N}$ the number of elements in S does not exceed the number of elements in the set $S_m \stackrel{def}{=} \bigcup_{i \in S} K^i_m$:

$$\#S \leq \#S_m$$

But it is not at all obvious that this condition is also **sufficient** for the existence of the injection in question, provided the sets K^i_m are **finite**. This assertion is known as the Hall-König Theorem, and B. Mityagin was the first to realize its relevance to the Quasi-equivalence Problem [14].

It is easy to show that the sets K^i_m are finite for a nuclear space E, actually what we need is not nuclearity, but a weaker property, namely, the fact that the space is a **Schwartz space**, which in case of Köthe spaces boils down to the condition

$$\lambda_{ip}/\lambda_{i,(p+1)} \to 0, \qquad \mu_{ip}/\mu_{i,(p+1)} \to 0, \qquad \text{as } i \to \infty$$

Obviously, every nuclear space is a Schwartz space. Consider the set

$$K^i(p, P+1; P, Q) = \{n \in \mathbb{N} : \frac{\lambda_{ip} \mu_{n,(P+1)}}{\mu_{nP} \lambda_{iQ}} \leq 1\}$$

This set is finite, since

$$\mu_{n,P}/\mu_{n,(P+1)} \to 0, \quad \text{as } n \to \infty$$

Taking $P = m(p)$, $q = P + 1$, $Q = m(m(p) + 1)$, we see, that K_m^i is a part of a finite set $K^i(p, P + 1; P, Q)$.

Let us formulate the result in a slightly different manner:

Two bases in question are **not** quasi-equivalent if for any function $m : \mathbb{N} \to \mathbb{N}$ there exists a finite set S such that

$$\#S > \#S_m = \# \bigcup_{i \in S} \bigcap_{p,q} K^i(p, q; m(p), m(q))$$

This means that for every pair $(i, n) \in S \times (\mathbb{N} \setminus S_m)$ there exists a pair $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that

$$(14) \qquad \frac{\lambda_{ip}\mu_{nq}}{\mu_{n,m(p)}\lambda_{i,m(q)}} > 1$$

Let

$$L_m(p, q) = \{(i, n) \in \mathbb{N} \times \mathbb{N} : \frac{\lambda_{ip}\mu_{nq}}{\mu_{n,m(p)}\lambda_{i,m(q)}} > 1\}$$

Then these sets cover the set $S \times (\mathbb{N} \setminus S_m)$:

$$\bigcup_{p,q} L_m(p, q) \supset S \times (\mathbb{N} \setminus S_m)$$

Let us remark that if $q \leq m(p)$ and $p \leq m(q)$ (we always assume that $p \leq m(p), q \leq m(q)$), then $K^i(p, q; m(p), m(q)) = \mathbb{N}$, so these sets are not interesting in our considerations. Therefore we always assume that either $q > m(p)$ or $p > m(q)$.

The fact that the systems $\{e_i, i = 1, 2, \dots\}$ and $\{f_i, i = 1, 2, \dots\}$ are absolute bases can be expressed by the following inequalities (recall (8), (9), (1)):

$$(15) \qquad \begin{array}{c} \|\Lambda_p \, A \, M_{p+1}^{-1}\|_{l_1 \to l_1} \leq 1/2 \\ \|M_p A^{-1} \, \Lambda_{p+1}^{-1}\|_{l_1 \to l_1} \leq 1/2 \end{array}$$

Therefore we readily obtain that

$$\|\Lambda_q \cdot (A_+(A^{-1})_+)^{m(q)-q-1} \cdot A_+ \cdot M_{m(q)}^{-1} M_p \cdot (A^{-1})_+ (A_+(A^{-1})_+)^{m(p)-p-1} \cdot \Lambda_{m(p)}^{-1}\|_{l_1 \to l_1}$$
$$\leq (1/2)^{2m(q)+2m(p)-2q-2p} = (1/4)^{m(q)+m(p)-q-p}$$

By Lemma 1, this condition allows to reconstruct the missing diagonal matrices.

Let $C = A_+(A^{-1})_+$. Obviously, C dominates the identity matrix. We can rewrite:

$$\|\Lambda_q C^{m(q)-q-1} A_+ M_{m(q)}^{-1} M_p (A^{-1})_+ C^{m(p)-p-1} \Lambda_{m(p)}^{-1}\|_{l_1 \to l_1}$$
$$\leq (1/4)^{m(q)+m(p)-q-p}$$

Further, for the spectral radius of this matrix, we have

$$\rho(\Lambda_{m(p)}^{-1}\Lambda_q \cdot C^{m(q)-q-1} \cdot A_+ \cdot M_{m(q)}^{-1} M_p \cdot (A^{-1})_+ \cdot C^{m(p)-p-1})$$
$$\leq \|\Lambda_{m(p)}\Lambda_{m(p)}^{-1}\Lambda_q C^{m(q)-q-1} A_+ M_{m(q)}^{-1} M_p (A^{-1})_+ C^{m(p)-p-1}\Lambda_{m(p)}^{-1}\|_{l_1 \to l_1}$$
$$\leq (1/4)^{m(q)+m(p)-q-p}$$

In relations (13) there appear only ratios $\dfrac{\lambda_{ip}}{\lambda_{n,m(q)}}$ so only these ratios really matter.

Therefore we actually may choose the sequence $\Lambda_{m(p)}$ arbitrarily, and so, by Lemma 2, the previous estimate is sharp.

Lemma 4. *Let $A = (\alpha_i^j)$ be a $n \times n$ matrix with non-negative entries. Then*

$$\forall i \qquad \alpha_i^i \leq \rho(A)$$

Proof.

Since the (i,i)-th element of A^n is not smaller than $(\alpha_i^i)^n$ for a non-negative matrix A, we get

$$\alpha_i^i \leq (\|A^n\|_{l_1 \to l_1})^{1/n} \to \rho(A) \qquad \blacksquare$$

Let us show that under some assumptions on the matrices $\Lambda_p, M_p, A, A^{-1}$ the relations (14), (15) are self contradictory, i.e., the related bases are quasi-equivalent.

We first give another proof to a theorem due to L. Crone and W. B. Robinson [1] and V.P. Kondakov [10].

An absolute basis $\{e_i, i = 1, 2, \dots\}$ is called **regular** if for any p the sequence

$$\frac{\lambda_{ip}}{\lambda_{i,p+1}}$$

is decreasing in i.

This notion was introduced by M.M. Dragilev [4]. It is known that if a Fréchet space has an absolute regular basis then one can rearrange any other absolute basis $\{f_i, i = 1, 2, \dots,\}$ so that it will become a regular basis (see, e.g., [5]).

Theorem (Crone-Robinson-Kondakov). *Let E be a Fréchet - Schwartz space with a **regular** absolute basis $\{e_i, i = 1, 2, \dots\}$. Then any absolute basis $\{f_i, i = 1, 2, \dots\}$ of this space is quasi-equivalent to the basis $\{e_i, i = 1, 2, \dots\}$.*

Proof.

Assume the opposite and let $\{f_i, i = 1, 2, \dots\}$ be an absolute basis, which is **not** quasi-equivalent to $\{e_i, i = 1, 2, \dots\}$. We may assume that this new basis is also regular. Because of the assumed non-quasi-equivalence of the bases, for every function $m : \mathbb{N} \to \mathbb{N}$ there exists a finite set $S \subset \mathbb{N}$ such that

$$\#S > \#S_m$$

Therefore the set $S \times (\mathbb{N} \setminus S_m)$ intersects the diagonal $\{(i,i) \in \mathbb{N} \times \mathbb{N}\}$. Let $(i_0, i_0) \in S \times (\mathbb{N} \setminus S_m)$.

Since the sets $L_m(p,q)$ cover the set $S \times (\mathbb{N} \setminus S_m)$ there exists a pair p_0, q_0 such that $(i_0, i_0) \in L_m(p_0, q_0)$,

Let us consider the case $m(q_0) > q_0 > m(p_0) > p_0$, (as for the remaining case $m(p_0) > p_0 > m(q_0) > q_0$, it can be treated in the same way).

Under this assumption the numbers $\dfrac{\lambda_{ip_0}}{\lambda_{i,m(q_0)}}$ are decreasing in i, and the numbers $\dfrac{\mu_{jq_0}}{\mu_{j,m(p_0)}}$ are increasing in j. So, for every $i \leq i_0$, and every $j \geq i_0$ we have

$$\frac{\lambda_{ip_0} \mu_{jq_0}}{\mu_{j,m(p_0)} \lambda_{i,m(q_0)}} \geq 1$$

Therefore the whole set $r = \{(i,j) : i \leq i_0 \leq j\}$ is covered by $L_m(p_0, q_0)$, i.e.,

$$r \subset L_m(p_0, q_0)$$

Let $q = \{(i,j): \ i \le i_0, \ j < i_0\}$.
Let $A_1 = (\alpha_i^j)_{(i,j)\in q}, \quad B_1 = (\beta_j^i)_{(i,j)\in q}, \quad A_2 = (\alpha_i^j)_{(i,j)\in r}, \quad B_2 = (\beta_j^i)_{(i,j)\in r}.$
On one hand, it is obvious that

$$A_1 B_1 + A_2 B_2 = I_{i_0}$$

where I_{i_0} is the identity matrix of the size $i_0 \times i_0$. The matrix $A_1 B_1$ is degenerate (A_1 is of the size $i_0 \times (i_0 - 1)$), so $I_{i_0} - A_2 B_2$ is degenerate, so $A_2 B_2$ has a fixed vector, therefore $1 \in Spec\, A_2 B_2$, and for the spectral radius of $A_2 B_2$ we have:

$$\rho(A_2 B_2) \ge 1$$

Let us show that, on the other hand, our assumptions imply that the spectral radius of $A_2 B_2$ is very small:
Let

$$\Lambda = \left(\sqrt{\frac{\lambda_{i,q_0}}{\lambda_{i,m(p_0)}}} \, \delta_i^j \right)_{i,j \le i_0}$$

$$M = \left(\sqrt{\frac{\mu_{i,p_0}}{\mu_{i,m(q_0)}}} \, \delta_i^j \right)_{i,j \ge i_0}$$

Then the matrix $\Lambda A_2 M$ is entrywise greater than the matrix A_2, and the matrix $M B_2 \Lambda$ is entrywise greater than the matrix B_2. Therefore, the following estimate holds for the spectral radii:

$$\rho(A_2 B_2) \le \rho(\Lambda A_2 M^2 B_2 \Lambda) = \rho(\Lambda^2 A_2 M^2 B_2)$$

$$\le \rho(\Lambda_{m(p_0)}^{-1} \Lambda_{q_0} A_+ M_{m(q_0)}^{-1} M_{p_0} (A^{-1})_+)$$

$$\le \rho(\Lambda_{m(p_0)}^{-1} \Lambda_{q_0} C^{m(q_0)-q_0-1} A_+ M_{m(q_0)}^{-1} M_{p_0} (A^{-1})_+ C^{m(p_0)-p_0-1})$$

$$\le (1/4)^{m(q_0)+m(p_0)-q_0-p_0} < 1$$

This contradiction completes the proof. ■

Theorem 3. *Let E be a* **nuclear** *Fréchet space with bases $\{e_i, i = 1, 2, \dots\}$ and $\{f_i, i = 1, 2, \dots\}$. Let, as usual, $e_i = \sum_j \alpha_i^j f_j$, $f_i = \sum_j \beta_i^j e_j$, and let A, $A^{-1} = B$ denote the related matrices $A = (\alpha_i^j)_{i,j=1}^\infty$, $B = (\beta_i^j)_{i,j=1}^\infty$. If these matrices satisfy the condition $B = M\, A^* \Lambda$, where M and Λ are some diagonal matrices, then the bases $\{e_i\}$ and $\{f_i\}$ are quasi-equivalent.*

Proof.
Assume the opposite - let the bases be non-quasi-equivalent. Then for every function $m : \mathbb{N} \to \mathbb{N}$ there exists a finite set $S \in \mathbb{N}$ such that

$$\#S > \#S_m$$

By the condition of the Theorem $B = M\, A^* \Lambda$. This means that an appropriate scaling will make $B = A^*$, i.e., the bases are orthogonal in some wider Hilbert space. We assume that this scaling is already done and

$$A^{-1} = B = A^*$$

Let $s = S \times S_m$, $t = S \times (\mathbb{N} \setminus S_m)$. Let

$$A_1 = (\alpha_i^j)_{(i,j) \in s}, \quad B_1 = (\beta_i^j)_{(j,i) \in s}, \quad A_2 = (\alpha_i^j)_{(i,j) \in t}, \quad B_2 = (\beta_i^j)_{(j,i) \in t}$$

Note that, according to our assumption, $B_2 = A_2^*$.

As before, using the obvious identity $A_1 B_1 + A_2 B_2 = I_{\#S}$ and the assumed inequality $\#S > \#S_m$, we get:

$$1 \leq \rho(A_2 B_2) = \rho(A_2 A_2^*)$$

Now we show that

$$1/4 \geq \rho(A_2 A_2^*)$$

which will conclude the proof.

Since (e_i) and (f_i) are (absolute) bases we have

$$\|\Lambda_p \, A \, M_{p+1}^{-1}\|_{l_1 \to l_1} \leq 1/2$$

and

$$\|M_p B \, \Lambda_{p+1}^{-1}\|_{l_1 \to l_1} \leq 1/2$$

Since E is assumed to be nuclear, we may replace the l_1-norms by the l_∞-norms and get

$$\|M_p B \, \Lambda_{p+1}^{-1}\|_{l_\infty \to l_\infty} \leq 1/2$$

which can be rewritten as

$$\|\Lambda_{p+1}^{-1} B^* \, M_p\|_{l_1 \to l_1} \leq 1/2$$

or

$$\|\Lambda_{p+1}^{-1} A \, M_p\|_{l_1 \to l_1} \leq 1/2$$

Therefore

$$\|\Lambda_p \, A \, M_{m(p)}^{-1}\|_{l_1 \to l_1} \leq (1/2)^{m(p)-p}$$

and

$$\|\Lambda_{m(p)}^{-1} A \, M_p\|_{l_1 \to l_1} \leq (1/2)^{m(p)-p}$$

This immediately implies that

$$\|\Lambda_p \, A_2 \, M_{m(p)}^{-1}\|_{l_1 \to l_1} \leq (1/2)^{m(p)-p}$$

and

$$\|\Lambda_{m(p)}^{-1} A_2 \, M_p\|_{l_1 \to l_1} \leq (1/2)^{m(p)-p}$$

(since $A_2 = (\alpha_i^j)_{(i,j) \in t}$ we restrict the diagonal matrices Λ_p to $\{(i,j) \in (\mathbb{N} \setminus S_m) \times (\mathbb{N} \setminus S_m)\}$ and M_p to $\{(i,j) \in S \times S\}$.

For every $(i,j) \in t$ there exist p, q such that

$$(i,j) \in L_m(p,q)$$

so

$$1 \leq \frac{\lambda_{ip} \mu_{jq}}{\mu_{j,m(p)} \lambda_{i,m(q)}}$$

therefore either $1 \leq \dfrac{\lambda_{ip}}{\mu_{j,m(p)}}$ or $1 \leq \dfrac{\mu_{jq}}{\lambda_{i,m(q)}}$. Anyway,

$$1 \leq \sum_q \frac{\mu_{jq}}{\lambda_{i,m(q)}} + \sum_p \frac{\lambda_{ip}}{\mu_{j,m(p)}}$$

Recalling that $A_2 = (\alpha_i^j)_{(i,j)\in t}$ and applying the Interpolation Lemma 3, we see that

$$\|A_2\|_{l_1 \to l_1} \leq \sum_q \|\Lambda_q\, A_2\, M_{m(q)}^{-1}\|_{l_1 \to l_1} + \sum_p \|\Lambda_{m(p)}^{-1} A_2\, M_p\|_{l_1 \to l_1}$$

$$\leq 2 \sum_p (1/2)^{m(p)-p}$$

Choosing the sequence $m(p), p = 1, 2, \dots$, sufficiently fast growing, we see that

$$\|A_2\|_{l_1 \to l_1} \leq 1/2$$

Replacing the l_1-norms by $l_\infty-$norms, we see that

$$\|A_2\|_{l_\infty \to l_\infty} \leq 1/2$$

or

$$\|A_2^*\|_{l_1 \to l_1} \leq 1/2$$

So we get:

$$\rho(A_2 A_2^*) \leq \|A_2 A_2^*\|_{l_1 \to l_1} \leq 1/4 \quad \blacksquare$$

Theorem 4. *Let E be a Fréchet - Schwartz space with absolute bases*

$$\{e_i, i = 1, 2, \dots\} \ \text{and} \ \{f_i, i = 1, 2, \dots\}$$

If the transformation matrices A and $A^{-1} = B$ are triangular then the bases $\{e_i\}$ and $\{f_i\}$ are quasi-equivalent.

Proof.
    Assume the opposite. Then, as earlier, we come to the conclusion that for any function $m : \ \mathbb{N} \to \mathbb{N}$ there exists a finite set S such that there exists

$$(i, i) \in S \times (\mathbb{N} \setminus S_m).$$

Note that $\alpha_i^i \beta_i^i = 1$ since the matrices A and B are triangular and mutually inverse. Since $(i, i) \in S \times (\mathbb{N} \setminus S_m)$, there exist p, q such that $(i, i) \in L_m(p, q)$. Therefore

$$|\alpha_i^i \beta_i^i| \leq \lambda_{ip} |\alpha_i^i| \mu_{i,m(p)}^{-1} \mu_{iq} |\beta_i^i| \lambda_{i,m(q)}^{-1}$$

$$\leq \sum_j \lambda_{ip} |\alpha_j^i| \mu_{j,m(p)}^{-1} \mu_{jq} |\beta_i^j| \lambda_{i,m(q)}^{-1}$$

$$= (\Lambda_p (A_2)_+ M_{m(p)}^{-1} M_q (B_2)_+ \Lambda_{m(q)}^{-1})_i^i$$

Applying Lemma 4, we see that

$$|\alpha_i^i \beta_i^i| \leq \rho(\Lambda_p(A_2)_+ M_{m(p)}^{-1} M_q(B_2)_+ \Lambda_{m(q)}^{-1})$$

$$\leq \|\Lambda_p(A_2)_+ M_{m(p)}^{-1} M_q(B_2)_+ \Lambda_{m(q)}^{-1}\|_{l_1 \to l_1}$$

$$\leq (1/4)^{m(p)+m(q)-p-q}$$

Choosing the sequence $m(p), p = 1, 2, ...,$ sufficiently fast growing, we come to a contradiction with the fact that $\alpha_i^i \beta_i^i = 1$.   ∎

Since "any" matrix can be represented as a product of an orthogonal and a triangular matrices, it is tempting to try to prove the Quasi-equivalence conjecture combining the above two results. In this direction we can prove the following result.

Let us first introduce a new notion:

Definition. *An absolute basis* $\{e_i, i = 1, 2, ...\}$ *in a Fréchet space* E *is called* **subregular** *if*

$$\forall p \qquad \exists P \qquad \sup_{i,j:j \geq i} \frac{]e_i[_p}{]e_j[_P} < \infty$$

Here $] \cdot [_p, p = 1, 2, ...,$ *is a fundamental system of seminorms in* E.

Obviously, a regular basis can be made subregular by appropriate scalings (forcing $]e_i[_1 = 1$, then $]e_i[_p = \dfrac{]e_i[_p}{]e_i[_1}$ is increasing in i – this guarantees subregularity). It is not difficult to present examples of subregular bases which are not regular (let $]e_i[_p$ be increasing in i, make $\dfrac{]e_i[_p}{]e_i[_q}$ very far from monotonic - this will destroy regularity).

Theorem 5. *Let* E *be a* **nuclear** *Fréchet space with a* **subregular** *basis* $\{e_i, i = 1, 2, ...\}$. *Let* $\{f_i, i = 1, 2, ...\}$ *be another basis in* E. *Assume that the following is true for the related transformation matrices* $A, B = A^{-1}$:

$$\|A\|_{l_1 \to l_2} < \infty, \qquad \|B\|_{l_1 \to l_2} < \infty$$

Then these bases are quasi-equivalent.

Proof.

Since $\|A\|_{l_1 \to l_2} < \infty$, the columns of the matrix A are vectors from l_2. Therefore we can apply the Schmidt orthogonalization procedure to the columns of the matrix A and thus represent it as a product of an upper triangular matrix T and an orthogonal matrix U :

$$A = UT, \qquad B = A^{-1} = T^{-1}U^{-1} = T^{-1}U^*$$

Let $T = (t_i^j)$ and $T^{-1} = (\tau_i^j)$. Then $t_j^i = \tau_j^i = 0$ for $j < i$. Because of the assumptions of the Theorem

$$\|T\|_{l_1 \to l_2} = \|U^{-1}A\|_{l_1 \to l_2} \leq \|U^{-1}\|_{l_2 \to l_2}\|A\|_{l_1 \to l_2} = \|A\|_{l_1 \to l_2} < \infty$$

Similarly,

$$\|T^{-1}\|_{l_1 \to l_2} < \infty$$

Therefore

$$|t_i^j| \le (\sum_k |t_i^k|^2)^{1/2} \le \|T\|_{l_1 \to l_2}$$

and

$$|\tau_i^j| \le \|T^{-1}\|_{l_1 \to l_2}$$

We will show that the system $g_i = \sum_j t_i^j e_j, i = 1, 2, ...$, forms an (absolute) basis in E. Then the bases $\{g_i\}$ and $\{e_i\}$ have triangular transformation matrices, and therefore they are quasi-equivalent, by Theorem 4. The bases $\{f_i\}$ and $\{g_i\}$ have orthogonal transformation matrices and they are quasi-equivalent by virtue of Theorem 3. So we will be able to prove the result.

By Theorem 1, we need to show that

$$\forall p \qquad \exists P \qquad \|\Lambda_p T_+ (T^{-1})_+ \Lambda_P^{-1}\|_{l_1 \to l_1} \le 1$$

or,

$$\sup_i \sum_j \lambda_{ip} \sum_k |t_k^i \tau_j^k| \lambda_{jP}^{-1} \le 1$$

Choose Q such that

$$\sup_{i,j:j \ge i} \frac{\lambda_{ip}}{\lambda_{j,Q-1}} \le 1$$

and

$$\sum_j \frac{\lambda_{j,Q-1}}{\lambda_{j,Q}} \le 1$$

This is possible because of the assumed subregularity and nuclearity. Then

$$\sup_i \sum_{j:j \ge i} \frac{\lambda_{ip}}{\lambda_{j,Q}} \le 1$$

and

$$\|\Lambda_p T_+ \Lambda_Q^{-1}\|_{l_\infty \to l_\infty} = \sup_i \sum_j \lambda_{ip} |t_j^i| \lambda_{jQ}^{-1}$$

$$= \sup_i \sum_{j:j \ge i} \lambda_{ip} |t_j^i| \lambda_{jQ}^{-1} \le \|T\|_{l_1 \to l_2}$$

Similarly, choose P such that

$$\sup_{i,j:j \ge i} \frac{\lambda_{iQ}}{\lambda_{j,P-1}} \le 1$$

and

$$\sum_j \frac{\lambda_{j,P-1}}{\lambda_{j,P}} \le 1$$

Then

$$\sup_i \sum_{j:j\geq i} \frac{\lambda_{iQ}}{\lambda_{j,p}} \leq 1$$

and

$$\|\Lambda_Q(T^{-1})_+ \Lambda_P^{-1}\|_{l_\infty \to l_\infty} = \sup_i \sum_j \lambda_{iQ} |\tau_j^i| \lambda_{jP}^{-1}$$

$$= \sup_i \sum_{j:j\geq i} \lambda_{iQ} |\tau_j^i| \lambda_{jP}^{-1} \leq \|T^{-1}\|_{l_1 \to l_2}$$

and therefore

$$\|\Lambda_p T_+(T^{-1})_+ \Lambda_P^{-1}\|_{l_\infty \to l_\infty}$$

$$\leq \|\Lambda_p T_+ \Lambda_Q^{-1}\|_{l_\infty \to l_\infty} \|\Lambda_Q(T^{-1})_+ \Lambda_P^{-1}\|_{l_\infty \to l_\infty} < \infty$$

and the result is completely proven. ∎

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