CONVEX GEOMETRY OF COXETER-INVARIANT POLYHEDRA

NICHOLAS MCCARTHY, DAVID OGILVIE, NAHUM ZOBIN,
AND VERONICA ZOBIN

ABSTRACT. We study the facial structure of convex polyhedra invariant under the natural action of a Coxeter group. The results are applied to the study of faces of maximal dimension of orbihedra related to some non-Coxeter groups.

1. INTRODUCTION

Let G be a finite Coxeter group naturally acting on a finite dimensional real space V. We study the geometry of convex G-invariant polyhedra.

The simplest convex G-invariant polyhedron is a G-**orbihedron** $\operatorname{Co}_G x$ – the convex hull of the G-orbit of x, $x \in V$. Geometric properties of G-orbihedra play important roles in many problems, ranging from Topology and Algebra to Operator Theory and Statistics– see, e.g., [1, 8, 9, 10, 12, 14, 17, 18, 19]. G-orbihedra may be viewed as building blocks of general G-invariant convex polyhedra – every such polyhedron may be represented as the convex hull of a finite number of G-orbihedra.

We study the facial structure of a convex G-invariant polyhedron. It is natural to start with faces of maximal dimension. One can always introduce a G-invariant bilinear symmetric positive definite form on V, so we may assume that V is Euclidean and that G is a subgroup of the orthogonal group. The most simple and fundamental geometric characteristic of such face is its normal vector which can be identified with an extreme vector of the polar polyhedron.

Normals to faces of maximal dimension for the simplest G-invariant polyhedron — a G-orbihedron — can be completely described in convenient geometric terms, see [19], or Section 3 below. Many deep problems require very precise understanding of both the generic structure of such normals and the sorts of degenerations that may occur when the vector x approaches special subsets of V. In the present paper we use this description to study the geometric structure of faces of all dimensions for G-orbihedra.

As it has been already mentioned, every G-invariant convex polyhedron can be represented as a convex hull of finitely many G-orbihedra. The minimal number of the required G-polyhedra is a natural measure of complexity of the G-invariant convex polyhedron. If this number is small (compared to the dimension of the space) then only vectors of very specific structure can serve as normals to faces of the maximal dimension. If this number is large (greater than or equal to the dimension of the space), then any nonzero vector can be a normal to a face of a G-invariant convex polyhedron.

As soon as we depart from the natural representation of a Coxeter group, the problem of description of the convex structure of the related orbihedra becomes much more difficult. For example, consider a Coxeter group G naturally acting on V, and let $G^2 = G \times G$ act on $V \otimes V$ in the usual tensor way:

$$(g_1, g_2)(v_1 \otimes v_2) = (g_1 v_1) \otimes (g_2 v_2).$$

Note that this action on $V \otimes V$ is not generated by reflections across hyperplanes. Nonetheless, G^2 is a Coxeter group, but its natural representation is on $V \bigoplus V$:

$$(g_1, g_2)(v_1 \oplus v_2) = (g_1 v_1) \oplus (g_2 v_2).$$

Preliminary computer experiments (C.K. Li, I. Spitkovsky and N. Zobin) show that the structure of normals to faces of the orbihedra related to the tensor action of G^2 may be quite wild even if $\dim V = 3$. This is not too surprising — see [2] for a study of closely related topics from the Complexity Theory viewpoint.

Nevertheless, for groups of operators close to Coxeter ones it is still possible to obtain rather detailed results concerning the geometric structure of the related orbihedra. Consider a finite group K of operators, acting on V. It may contain reflections across hyperplanes, so consider the subgroup G generated by all such reflections in K. Assume that G acts effectively (i.e., without nontrivial fixed vectors) on V, so G is a Coxeter subgroup. The description of K-orbihedra can be reduced to a description of G-invariant convex polyhedra. If the index of G in K is small compared to the dimension of V (in this case K should be called a **quasi-Coxeter** group) then we can use the Coxeter machinery, which

makes it possible to describe the geometric structure of K-orbihedra. In particular, we describe the normals to faces of maximal dimension for $S_2(G)$-orbihedra, where S_2 is the group of permutations of $\{1, 2\}$, G is a Coxeter group acting on V, and the group $S_2(G) = S_2 \times G^2$ acts on the space $V^2 = V \bigoplus V$ as follows:

$$(\sigma, g_1, g_2)(v_1 \oplus v_2) = (g_1 v_{\sigma(1)}) \oplus (g_2 v_{\sigma(2)}).$$

Actually, it was this problem that stimulated the whole project. Group $S_2(G)$ has an index 2 Coxeter subgroup G^2, so K is quasi-Coxeter. In the case when $G = B_2$ this problem was studied and solved by the last two authors (see [21]), using vastly different methods which seemingly cannot be extended even to B_m with greater m. Though the present paper is completely independent of [21] the results and ideas from [21] were very helpful to us. In particular, the idea of consideration of a Coxeter subgroup already appeared in [21] though played there a rather technical role.

The paper is organized as follows: Section 2 contains a brief introduction to Coxeter groups adjusted to our needs, in Section 3 we present old and new results concerning the structure of normals to the faces of G-orbihedra of maximal dimension. We describe the faces of maximal dimension adjacent to a given vertex of a G-orbihedron, and as a corollary obtain some known results about simplicial orbihedra. In Section 4 we complement results of the previous section by a description of faces of lower dimensions. Section 5 is devoted to description of faces of maximal dimension for general G-invariant convex polyhedra. We also briefly discuss applications of these results to some problems of linear algebra. In Section 6 we apply these results to investigate the geometric structure of K-orbihedra for quasi-Coxeter groups K, and in particular, for the group $S_2(G)$. Section 7 contains a brief introduction to the duality approach to Operator Interpolation, its goal is to explain why the geometric results of the preceding sections are important in this field.

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2. A Brief Review of Coxeter Groups

Let us address several facts concerning the theory of Coxeter groups. For greater detail, consult [3], [5], or [11]. Let G be a group of linear operators on a finite dimensional real space V. Then G is called a **Coxeter group** if it is finite, generated by reflections across hyperplanes, and acts **effectively** (i.e., if $gx = x$ for all $g \in G$ then $x = 0$).

Again, one can always introduce a G-invariant bilinear symmetric positive definite form on V, turning V into a Euclidean space, and making all operators from G orthogonal. So we always assume that this has already been done. By definition, a Coxeter group is a group of linear operators, so it has a preferred representation which is called the **natural representation** or the **natural action**. One can describe Coxeter groups in pure group-theoretic terms, namely, in terms of generators and relations, see, e.g., [5].

2.1. **Roots and weights.** Consider the set $\mathcal{M}_G$ of all **mirrors** — hyperplanes H such that the orthogonal reflection across H belongs to G. These mirrors divide V into connected components, each one a simplicial cone. The closures of these cones are called **Weyl chambers** of G. Weyl chambers are **fundamental domains** of G, i.e., every G-**orbit** $\mathrm{Orb}_G\, x = \{gx : g \in G\}$ intersects every Weyl chamber at exactly one point, let this point be denoted by $x^* = x^*(C, G)$.

Fix a Weyl chamber C. A **wall** of C is a $(\dim V - 1)$-dimensional face of C, contained in a mirror. Reflections across the walls of C (i.e., across the related mirrors) generate the whole group G. The finiteness of G implies that the angle between any two walls of C must equal π/k for some natural $k \geq 2$.

For every wall W_i of C, let n_i be the **root** – a specially scaled normal vector to W_i pointing inwards with respect to C. It is convenient for us to choose all roots to be unit vectors (the standard normalization of roots is different, see, e.g., [5]). Since C is a simplicial cone, for each wall W_j there exists a unique extreme ray of C not lying on W_j. Let ω_j be a vector pointing in the direction of this extreme ray, so that $\langle n_i, \omega_j \rangle = c_j \delta_{ij}$, $c_j > 0$. Each ω_j is called a **fundamental weight** of G. Note that we prefer not to normalize fundamental weights, for the standard normalization see [5]. Let $\mathcal{R}_G$ and $\mathcal{W}_G$ denote the sets of all roots and, respectively, the set of all weights of G (i.e., associated with all Weyl chambers). Since group G acts **(simply) transitively** on the set of its Weyl chambers (i.e., for any two Weyl chambers C_1, C_2 there exists (exactly one) $g \in G$ such that $gC_1 = C_2$), then

$$\mathcal{R}_G = \bigcup_i \mathrm{Orb}_G\, n_i, \ \ \mathcal{W}_G = \bigcup_i \mathrm{Orb}_G\, \omega_i.$$

2.2. **Coxeter graphs.** There is a graph $\Gamma(G)$ (called the **Coxeter graph**) assigned to each Coxeter group. Fix a Weyl chamber C. The set $\mathrm{ver}\,(G)$ of vertices of the graph is in a one-to-one correspondence with the set of walls of C. Two vertices of this graph are connected with an edge if and only if the angle between the related walls is $\pi/k, k \geq 3$. The number $k - 2$ is the multiplicity of this edge. Obviously, the Coxeter graph does not depend upon the choice of the Weyl chamber.

In particular, every wall (but not the mirror containing this wall !) of any Weyl chamber is associated with a vertex of $\Gamma(G)$, and walls transformed one into another by the action of G are associated with the same vertex. Similarly, each weight ω is associated with a vertex $\pi(\omega)$ of the Coxeter graph $\Gamma(G)$. Obviously, $\pi(g\omega) = \pi(\omega)$ for every $g \in G$. So, $\pi(\omega)$ actually depends only upon the G-orbit of ω. Every vertex from ver (G) determines the G-orbit of exactly one weight (up to a positive factor).

A Coxeter group is irreducible if and only if its Coxeter graph is connected.

Notably, a Coxeter graph completely determines its Coxeter group, so if Γ is a Coxeter graph, let $G(\Gamma)$ denote the related Coxeter group.

There exists a full classification of connected Coxeter graphs, which implies a full classification of irreducible Coxeter groups. It worth noting that a reducible Coxeter group G is naturally isomorphic to the direct product of irreducible Coxeter groups $G(j)$ whose Coxeter graphs are the components j of $\Gamma(G)$, independently acting on mutually orthogonal subspaces $V(j)$. Let $J(G)$ denote the set of components of $\Gamma(G)$. Then

$$G = \prod_{j \in J(G)} G(j), \ \ V = \bigoplus_{j \in J(G)} V(j),$$

and if

$$g = (g(j))_{j \in J(G)} \in \prod_{j \in J(G)} G(j), \ v = \bigoplus_{j \in J(G)} v(j) \in \bigoplus_{j \in J(G)} V(j),$$

then

$$gv = \bigoplus_{j \in J(G)} g(j)v(j) \in \bigoplus_{j \in J(G)} V(j).$$

2.3. **Supports and stabilizers.** Fix a Weyl chamber C, let $x^* = x^*(C, G)$ be the unique vector in $C \cap \mathrm{Orb}_G x$. Since $x^* \in C$ and C is a simplicial cone, then there exists a unique decomposition of x^* into a positive linear combination of the related fundamental weights:

$$x^* = \sum_i \lambda_i \omega_i, \quad \lambda_i \geq 0.$$

Let us introduce the **support** of x as follows:

$$\mathrm{supp}_G x = \{\pi_i \in \mathrm{ver}\,(G) : \lambda_i > 0\}.$$

In other words, a vertex π_i of the Coxeter graph $\Gamma(G)$ belongs to $\mathrm{supp}_G x$ if x^* does not belong to the related wall W_i. One can easily show that $\mathrm{supp}_G x$ does not depend upon the choice of the Weyl chamber C. In fact, $\mathrm{supp}_G x$ depends only upon the G-orbit of x, therefore

the notation $\operatorname{supp}_G i$ is meaningful for a G-orbit i. Note that

$$\operatorname{supp}_G x = \emptyset \quad \text{if and only if} \quad x = 0.$$

Let

$$J(G, x) = \{j \in J(G) : \operatorname{supp}_G x \text{ intersects } j\}.$$

Combining the definition of $\operatorname{supp}_G x$ with the description of the action of a reducible Coxeter group we see that

$$\operatorname{span}(\operatorname{Orb}_G x) = \bigoplus_{j \in J(G,x)} V(j).$$

Now let B be an arbitrary subset of V. Define

$$\operatorname{supp}_G B = \bigcup_{x \in B} \operatorname{supp}_G x.$$

In particular, we shall need the **carrier set** of a G-invariant convex polyhedron U which we define as

$$\operatorname{Carr}_G U = \operatorname{supp}_G \operatorname{Extr} U,$$

where $\operatorname{Extr} U$ denotes the set of extreme vectors (= vertices) of the polyhedron U.

For a subset $A \subset V$ consider the **stabilizer subgroup**

$$\operatorname{Stab}_G A = \{g \in G : \forall\, x \in A \quad gx = x\}.$$

This subgroup is generated by reflections across the mirrors containing A. It has only obvious fixed vectors, namely those in V^A — the intersection of all mirrors containing A. If A is not contained in any mirror then we put $V^A = V$. The orthogonal complement of V^A is obviously $(\operatorname{Stab}_G A)$-invariant. If we restrict the action of the subgroup $\operatorname{Stab}_G A$ to the subspace $V_A = (V^A)^\perp$, it will act there effectively, and therefore it will become a Coxeter group on V_A. Let G_A denote this Coxeter group:

$$G_A = \operatorname{Stab}_G A|_{V_A}.$$

Let proj_A, proj^A denote the orthogonal projectors onto V_A, V^A, respectively. Obviously, $\mathbf{I} = \operatorname{proj}_A + \operatorname{proj}^A$, $\operatorname{proj}_A \operatorname{proj}^A = \operatorname{proj}^A \operatorname{proj}_A = 0$.

There exists a useful connection between the orthogonal projector proj^A and the stabilizer subgroup $\operatorname{Stab}_G A$. For any finite group K of linear operators acting on V consider the K-**averaging** operator

$$\operatorname{av}_K = (1/\operatorname{card} K) \sum_{g \in K} g.$$

One can easily show that the range of the K-averaging operator is exactly the set of fixed vectors of K.

Lemma 2.1. *Let G be a Coxeter group. Then*

$$\operatorname{proj}^A = \operatorname{av}_{\operatorname{Stab}_G A}.$$

Proof. For $x \in V^A$ it is obvious that $\operatorname{av}_{\operatorname{Stab}_G A} x = x$. Since every vector in the range of $\operatorname{av}_{\operatorname{Stab}_G A}|_{V_A} = \operatorname{av}_{G_A}$ is obviously fixed by the action of $\operatorname{Stab}_G A$, and since $\operatorname{Stab}_G A$ acts effectively on V_A, then $\operatorname{av}_{\operatorname{Stab}_G A}|_{V_A} = 0$, which proves the Lemma. $\square$

Corollary 2.2. *Let U be a convex G-invariant set. Then*

$$\operatorname{proj}^A U = U \cap V^A.$$

Lemma 2.3. *Let G be a Coxeter group. The origin is a relatively interior point of $\operatorname{Co}_G x$ for every nonzero $x \in V$.*

Proof. Assuming that 0 is not in the relative interior of $\operatorname{Co}_G x$, we find a nonzero vector $b \in \operatorname{span} \operatorname{Orb}_G x$ such that $\langle b, gx \rangle \geq 0$ for all $g \in G$. Since G acts effectively then, by Lemma 2.1, $\operatorname{av}_G = 0$, so $0 = \operatorname{av}_G x = (1/\operatorname{card} G) \sum_{g \in G} gx$, and then $\langle b, gx \rangle = 0$ for all $g \in G$. This means that $b \perp \operatorname{Orb}_G x$, and since $b \in \operatorname{span} \operatorname{Orb}_G x$, we conclude that $b = 0$, contrary to the assumption. $\square$

Corollary 2.4. *Let G be an irreducible Coxeter group. The origin is an interior point of $\operatorname{Co}_G x$ for every nonzero $x \in V$.*

Lemma 2.5. *Let G be a Coxeter group. Then*

$$\operatorname{Co}_{\operatorname{Stab}_G A} x = \operatorname{proj}^A x + \operatorname{Co}_{G_A} \operatorname{proj}_A x.$$

In particular, $\operatorname{Co}_{\operatorname{Stab}_G A} x$ is a polyhedron in an affine plane of dimension $\leq \dim V_A$.

Let κ be a subset of the set $\operatorname{ver}(G)$ of vertices of the graph $\Gamma(G)$. Let $\Gamma(G) \setminus \kappa$ denote the graph obtained from $\Gamma(G)$ by erasing the vertices from κ together with the edges adjacent to these vertices.

The following three useful results follow almost immediately from the definitions and the above mentioned facts.

Lemma 2.6.

$$\Gamma(G_A) = \Gamma(G) \setminus \operatorname{supp}_G A.$$

Lemma 2.7.

$$\operatorname{supp}_{G_A} \operatorname{proj}_A x = \operatorname{supp}_G x \setminus \operatorname{supp}_G A.$$

Corollary 2.8.

$$\mathcal{W}_{G_A} = (\operatorname{proj}_A \mathcal{W}_G) \setminus \{0\}.$$

Let
$$m_G(x, y) = \sup\{\langle gx, y \rangle : g \in G\}.$$
Obviously, $m_G(x, y)$ depends only upon the G-orbits of x and y, so the notation $m_G(x, i)$ is meaningful for $x \in V$ and i a G-orbit, and for x and i both G-orbits.

Lemma 2.9 ([19]). *Vectors x and y belong to the same Weyl chamber if and only if*
$$m_G(x, y) = \langle x, y \rangle.$$
Moreover, in this case
$$\{z \in \operatorname{Orb}_G x : \langle z, y \rangle = m_G(x, y)\} = \operatorname{Orb}_{\operatorname{Stab}_G y} x.$$

One can easily deduce from Lemma 2.9 that $m_G(x, y) \geq 0$, and $m_G(x, y) = 0$ if and only if there are no components of $\Gamma(G)$ intersecting both $\operatorname{supp}_G x$ and $\operatorname{supp}_G y$. In particular, $m_G(x, y) > 0$ for an irreducible group G and nonzero x and y.

3. Convex Structure of Coxeter Orbihedra

3.1. **Polyhedra of full dimension and their polars.** Let G be a finite Coxeter group, maybe reducible. An orbihedron $\operatorname{Co}_G x$ is **of full dimension** (i.e., is not a subset in a proper subspace) if and only if x is not in a proper G-invariant subspace. This happens if and only if $J(G, x) = J(G)$, i.e., $\operatorname{supp}_G x$ intersects every component of $\Gamma(G)$. Let us agree that if $\operatorname{Co}_G x$ is not of full dimension then we regard it as a polyhedron in the subspace span $\operatorname{Orb}_G x = \bigoplus_{j \in J(G,x)} V(j)$, and we consider its faces of maximal dimension (= of codimension 1) in this subspace. This means that we actually consider the orbihedron with respect to the group
$$G[x] = G|_{\operatorname{span}(\operatorname{Orb}_G x)}.$$
Let $V^{[x]} = V^{\operatorname{Orb}_G x} = \operatorname{span}(\operatorname{Orb}_G x)$. Obviously, $\operatorname{Co}_G x = \operatorname{Co}_{(G[x])} x$ is of full dimension in $V^{[x]}$. Also, one can easily see that $\Gamma(G[x])$ is the disjoint union of the components of $\Gamma(G)$, intersecting with $\operatorname{supp}_G x$:
$$\Gamma(G[x]) = \bigsqcup \{j : j \in J(G, x)\}.$$
The reason for our desire to consider only polyhedra of full dimension is explained in the next paragraph.

For a subset $U \subset V$ let
$$U^\circ = \{y \in V : \forall \, x \in U \ \langle x, y \rangle \leq 1\}.$$
U° is called the **polar set** of U, it is convex, closed and contains the origin. Obviously, $U^\circ = (\operatorname{conv} U)^\circ$, where conv U denotes the convex hull of U. If U contains the origin, then, by the Bipolar Theorem, $(U^\circ)^\circ$

is the closed convex hull of U. If U is a G-invariant convex polyhedron for a Coxeter group G then U contains the origin and therefore

$$U = (U^\circ)^\circ = \{y \in V : \forall\ z \in U^\circ\ \langle y, z \rangle \leq 1\},$$

so we get a description of the polyhedron U in terms of linear inequalities. It is possible to switch to the smallest possible set of inequalities in this description. If the G-invariant convex polyhedron U is of full dimension, then, by Lemma 2.3 it contains the origin as an interior point and therefore its polar set U° is a compact polyhedron. So, by the Krein–Milman Theorem, it is the convex hull of the set $\mathrm{Extr}\,(U^\circ)$ of its extreme vectors,

$$U = (U^\circ)^\circ = (\mathrm{Extr}\,(U^\circ))^\circ = \{y \in V : \forall\ z \in \mathrm{Extr}\,(U^\circ)\ \langle y, z \rangle \leq 1\},$$

and this is obviously the smallest possible set of linear inequalities describing the polyhedron U. Affine hyperplanes $\{y \in V : \langle y, z \rangle = 1\}$, $z \in \mathrm{Extr}\,(U^\circ)$, carry codimension 1 faces of U, so the set $\mathrm{Extr}\,(U^\circ)$ is the set of normals to faces of U of codimension 1.

Let us note that if $\Gamma(G[x])$ is not connected (= if group $G[x]$ is reducible) then every G-orbihedron has a natural product structure:

$$\mathrm{Co}_{\,G}\, x = \prod_{j \in J(G,x)} \mathrm{Co}_{\,G(j)}\, \mathrm{proj}\,(j) x,$$

where $G(j)$ denotes the irreducible Coxeter group whose graph is $j \in J(G, x)$, $\mathrm{proj}\,(j)$ denotes the orthogonal projection onto the subspace $V(j)$ where $G(j)$ naturally acts.

3.2. **Orbihedra — faces of codimension** 1. The following result is an immediate corollary of Lemma 2.9.

Lemma 3.1. *If $y \in \mathrm{Extr}\,(\mathrm{Co}_{\,G}\, x)^\circ$ then the codimension 1 face*

$$\Phi(y) = \{z \in \mathrm{Co}_{\,G}\, x : \langle z, y \rangle = 1\}$$

of $\mathrm{Co}_{\,G}\, x$ coincides with $\mathrm{Co}_{\,\mathrm{Stab}_{\,G}\, y}\, g_0 x$, where $g_0 x$ is any vector from $\mathrm{Orb}_{\,G}\, x$ belonging to this face.

Now we can describe the set $\mathrm{Extr}\,(\mathrm{Co}_{\,G}\, x)^\circ$.

Theorem 3.2 ([19])**.** *Let G be a Coxeter group naturally acting on V. Then*

$$\mathrm{Extr}\,(\mathrm{Co}_{\,G}\, x)^\circ = \{y/m_G(x, y) \in V :\ \mathrm{supp}_{\,G}\, y\ \textit{consists of one vertex,}$$

$$\textit{belonging to}\ \Gamma(G[x]),\quad \textit{and}\quad \mathrm{supp}_{\,G}\, x\ \textit{intersects every component of}$$

$$\Gamma(G[x]) \setminus \mathrm{supp}_{\,G}\, y\}$$

Proof. It is easy to see that $z \in \mathrm{Extr}\,(\mathrm{Co}_G x)^\circ$ if and only if the set $\{gx : \langle gx, z \rangle = m_G(x, z) = 1\}$ spans the whole space $V^{[x]} = \mathrm{span}\,\mathrm{Orb}_G x$. Recall that by Lemma 3.1, $\{gx : \langle gx, z \rangle = m_G(x, z) = 1\} = \mathrm{Orb}_{\mathrm{Stab}_G z}\,g_0 x$, $\langle g_0 x, z \rangle = m_G(x, z) = 1$. By Lemma 2.5,

$$\mathrm{Co}_{\mathrm{Stab}_G z}\,g_0 x = \mathrm{proj}^z g_0 x + \mathrm{Co}_{G_z}\,\mathrm{proj}_z g_0 x,$$

so $z \in \mathrm{Extr}\,(\mathrm{Co}_G x)^\circ$ if and only if $\mathrm{Co}_{G_z}\,\mathrm{proj}_z g_0 x$ is of dimension $\dim V^{[x]} - 1$. This happens if and only if $\Gamma((G[x])_z)$ has exactly $\dim(V^{[x]}) - 1$ vertices. Since, by Lemma 2.7, $\Gamma((G[x])_z) = \Gamma(G[x]) \setminus \mathrm{supp}_G z$, this can happen if and only if $\mathrm{supp}_G z$ consists of one vertex, belonging to $\Gamma(G[x])$, and $\mathrm{supp}_{(G[x])_z}\,\mathrm{proj}_z g_0 x$ intersects every component of the graph $\Gamma(\mathrm{Stab}_{G[x]} z)$. By Lemma 2.7, $\mathrm{supp}_{(G[x])_z}\,\mathrm{proj}_z g_0 x = \mathrm{supp}_G x \setminus \mathrm{supp}_G z$, and by Lemma 2.6, $\Gamma(\mathrm{Stab}_{G[x]} z) = \Gamma(G[x]) \setminus \mathrm{supp}_G z)$. The Theorem is proven. $\square$

Since the only vectors having one-vertex supports are weights, and since $m_G(x, \omega) > 0$ for $\omega \in \mathcal{W}_{G[x]} = \mathcal{W}_G \cap V^{[x]}$, we see that $z \in \mathrm{Extr}\,(\mathrm{Co}_G x)^\circ$ if and only if

$$z = \omega/m_G(x, \omega), \ \ \omega \in \mathcal{W}_G, \ \ \pi(\omega) \in \Gamma(G[x]),$$

and

$$\mathrm{supp}_G x \ \text{ intersects every component of } \ \Gamma(G[x]) \setminus \{\pi(\omega)\}.$$

According to our agreement, we disregard all weights ω such that $\pi(\omega) \notin \Gamma(G[x])$, i.e., such that $m_G(x, \omega) = 0$.

Combining the previous results, we arrive to the following description.

Theorem 3.3. *Let G be a Coxeter group naturally acting in V. For every codimension 1 face Φ of $\mathrm{Co}_G x$ there exists a unique vector $\omega = \omega(\Phi) \in \mathcal{W}_{G[x]}$, such that:*

(i) $\mathrm{supp}_G x$ intersects every component of $\Gamma(G[x]) \setminus \mathrm{supp}_G \omega$,

(ii) $\Phi = \mathrm{Co}_{\mathrm{Stab}_G \omega}\,g_0 x$, where $g_0 \in G$ is such that $g_0 x$ and ω belong to one Weyl chamber.

Moreover, for every $\omega \in \mathcal{W}_{G[x]}$, satisfying (i), the set Φ defined in (ii) is a codimension 1 face of $\mathrm{Co}_G x$.

Corollary 3.4. *Let G be an irreducible Coxeter group. Let ω be a weight such that $\pi(\omega)$ is an end vertex of $\Gamma(G)$. Then $\omega \notin \mathrm{Extr}\,(\mathrm{Co}_G x)^\circ$ if and only if $\mathrm{supp}_G x = \mathrm{supp}_G \omega$.*

Using the remark preceding Lemma 3.1, we conclude that a face Φ of $\mathrm{Co}_G x$ has a natural product structure if the graph $\Gamma(G) \setminus \{\pi(\omega(\Phi))\}$ is not connected. So, if group G is irreducible, then the only faces Φ of $\mathrm{Co}_G x$ not having the natural product structure are those for which $\pi(\omega(\Phi))$ is an end vertex of $\Gamma(G)$.

3.3. **Counting vertices of orbihedra.** It is not difficult to find $\operatorname{card}_G x$ — the number of vertices in $\operatorname{Co}_G x$ (= the number of distinct vectors in $\operatorname{Orb}_G x$). It follows from the definition of the stabilizer subgroup that

$$\operatorname{card}_G x = \frac{\operatorname{card} G}{\operatorname{card} \operatorname{Stab}_G x} = \frac{\operatorname{card} G}{\operatorname{card} G_x}.$$

For every irreducible Coxeter group G the number $\operatorname{card} G$ is well known and may be found, e.g., in [5]. For a reducible group G we know that $G = \prod_{j \in J(G)} G(j)$ where $J(G)$ is the set of components of $\Gamma(G)$ and $G(j)$ denotes the irreducible Coxeter group whose graph is the component j. Therefore,

$$\operatorname{card} G = \prod_{j \in J(G)} \operatorname{card} G(j).$$

Since $\Gamma(G_x) = \Gamma(G) \setminus \operatorname{supp}_G x$, we can compute the number $\operatorname{card}_G x$ in convenient geometric terms.

Since $\operatorname{Co}_G x$ is of full dimension in $V^{[x]}$, then

$$\operatorname{card}_G x \geq 1 + \dim V^{[x]} = 1 + \operatorname{card} \operatorname{ver}(G_x).$$

So, for an irreducible group G we have $\operatorname{card}_G x \geq 1 + \dim V$.

Lemma 3.5. *Let G be an irreducible Coxeter group. Then $\operatorname{card}_G x = 1 + \dim V$ if and only if $G = A_n$ and $\operatorname{supp}_G x$ is an end vertex of $\Gamma(G)$.*

Proof. The "if" part can be verified directly: the orbit of the vector $(1, 1, \cdots, 1, -n)$ in the n-dimensional subspace $\{(x_1, x_2, \cdots, x_{n+1}) \in \mathbf{R}^{n+1} : \sum_i x_i = 0\}$ under the action of permutations consists of exactly $(n+1)$ vectors. Let us concentrate on the "only if" part. If $\operatorname{card}_G x = 1 + \dim V$, then $\operatorname{Co}_G x$ is a simplex in V, so $(\operatorname{Co}_G x)^\circ$ is also a simplex, so the set $\operatorname{Extr}(\operatorname{Co}_G x)^\circ$ consists of $(1 + \dim V)$ vectors, therefore it contains exactly one G-orbit. But $\operatorname{Extr}(\operatorname{Co}_G x)^\circ$ always contains the orbit of a weight associated with an end vertex of the Coxeter graph, therefore, $(\operatorname{Co}_G x)^\circ = \operatorname{Co}_G \omega$, $\operatorname{supp}_G \omega$ is an end vertex . Since $\operatorname{Co}_G x = (\operatorname{Co}_G \omega)^\circ$ and the latter is a simplex, we conclude that $\operatorname{supp}_G x$ is also an end vertex, and, besides, $\Gamma(G)$ has exactly two end vertices. Next, the group G cannot contain $-\mathbf{I}$ because a simplex cannot be central symmetric. It is known from the classification (see [5]) that A_n is the only irreducible Coxeter group possessing all these properties. $\qquad\square$

3.4. **Counting faces adjacent to a vertex.** Let us obtain a more explicit description of the faces of $\operatorname{Co}_G x$ adjacent to the vertex x. We may assume that $\operatorname{supp}_G x$ intersects all components of $\Gamma(G)$, so the G-orbihedron $\operatorname{Co}_G x$ is of full dimension.

Fix a Weyl chamber C, assume that $x \in C$. Let $\omega_j, \; j = 1, 2, \cdots, \dim V$, be the fundamental weights belonging to C. Then, by Theorem 3.2 the extreme vectors of $(\operatorname{Co}_G x)^\circ$, associated with the faces of maximal dimension adjacent to x are precisely those from the $\operatorname{Stab}_G x$-orbits of the vectors $\omega_j / m_G(x, \omega_j)$ such that $\operatorname{supp}_G x$ intersects every component of $\Gamma(G) \setminus \{\pi(\omega_j)\}$.

Let us introduce some terminology. We fix vector x, all forthcoming notions depend upon x. We say that a vertex π of $\Gamma(G)$ is **admissible** (more precisely, it should be called x-admissible, but we shall skip the label x if it does not create ambiguity) if $\operatorname{supp}_G x$ intersects every component of $\Gamma(G) \setminus \{\pi\}$. So, a vertex π of $\Gamma(G)$ is admissible if and only if the related vector $\omega_j / m_G(x, \omega_j), \; \pi = \pi(\omega_j)$, defines a codimension 1 face of $\operatorname{Co}_G x$ adjacent to x. The number of vectors in the $(\operatorname{Stab}_G x)$-orbit of ω_j is called the **multiplicity** of the admissible vertex $\pi = \pi(\omega_j)$. So, the sum of multiplicities of all admissible vertices gives the number of codimension 1 faces of $\operatorname{Co}_G x$ adjacent to x. We shall list all admissible vertices, together with their multiplicities, in convenient geometric terms.

Lemma 3.6. *The multiplicity of an admissible vertex from* $\operatorname{supp}_G x$ *is equal to* 1. *The multiplicity of an admissible vertex* π *belonging to a component* γ *of* $\Gamma(G) \setminus \operatorname{supp}_G x$ *is equal to* $1 + (\operatorname{card} G(\gamma) / \operatorname{card} G(\gamma \setminus \{\pi\}))$.

A vertex $\pi \in \operatorname{supp}_G x$ is called **interior** (or, better, x-interior) if it is adjacent only to vertices from $\operatorname{supp}_G x$. All non-interior vertices of $\operatorname{supp}_G x$ are called **boundary**.

Lemma 3.7. *Every interior vertex of* $\operatorname{supp}_G x$ *is admissible.*

Let γ be a component of $\Gamma(G) \setminus \operatorname{supp}_G x$. All end vertices of $\Gamma(G)$, belonging to γ, are called the **principal** vertices of γ.

Lemma 3.8. *Every principal vertex* π *of a component* γ *of* $\Gamma(G) \setminus \operatorname{supp}_G x$ *is admissible.*

Since $\operatorname{supp}_G x$ intersects every component of $\Gamma(G)$ then for every component of $\Gamma(G) \setminus \operatorname{supp}_G x$ there exists a vertex from $\operatorname{supp}_G x$ adjacent to this component. We say that γ is **acceptable** (or, better, x-acceptable) if there is exactly one vertex from $\operatorname{supp}_G x$ adjacent to γ.

Lemma 3.9. *Every acceptable component must contain at least one principal vertex.*

Proof. Indeed, if to assume the opposite, then each end vertex π of γ is not an end vertex of $\Gamma(G)$. The graph γ must have end vertices, since a

Coxeter graph cannot have cycles — see [5]. Therefore it must be adjacent to at least two other vertices of $\Gamma(G)$, but since π is an end vertex of γ, then at most one of these neighboring vertices is in γ, so at least one of them is in $\operatorname{supp}_G x$. Since γ is acceptable, then exactly one of the neighboring vertices is in $\operatorname{supp}_G x$. So, there are exactly two neighboring vertices, therefore π is adjacent to another vertex of γ. Therefore γ must have another end vertex (again, no cycles!). Repeating the same argument, we find another vertex from $\operatorname{supp}_G x$ adjacent to γ, which contradicts the acceptability. $\qquad\square$

So, the set of end vertices of an acceptable component consists of principal vertices plus, maybe, one non-principal vertex adjacent to $\operatorname{supp}_G x$. All end vertices of an acceptable component are principal if and only if the component consists of one principal vertex.

Corollary 3.10. *The only admissible vertices of an acceptable component are the principal vertices.*

Corollary 3.11. *The only non-admissible boundary vertices are those adjacent to at least one acceptable component.*

Lemma 3.12. *A non-acceptable component γ either contains no principal vertices or contains exactly one principal vertex and one branching vertex of $\Gamma(G)$.*

Proof. By the definition of a non-acceptable component, there exist at least two vertices from $\operatorname{supp}_G x$, adjacent to γ. This means that there are two options:

(1) there are at least two end vertices π_1, π_2 of γ adjacent to $\operatorname{supp}_G x$,

(2) there is an end vertex π of γ adjacent to at least two vertices of $\operatorname{supp}_G x$.

Consider the first option. Connected graph γ has at least two end vertices, therefore for every end vertex of γ there exists another vertex of γ adjacent to it. Therefore π_1, π_2 are not principal. If γ contains a principal vertex then γ has at least three end vertices. But a connected Coxeter graph cannot have more than three end vertices, so γ has exactly three end vertices, including exactly one principal vertex. Since γ has three end vertices, then, according to the classification of connected Coxeter graphs, it must have a branching vertex. Also, according to the classification, a connected Coxeter graph cannot have more than one branching vertex, so the statement is true in this situation.

Consider the second option. The vertex π is not principal. Therefore, if γ contains a principal vertex, then this principal vertex is an end vertex of γ, different from π. Since γ is connected, then π must be adjacent to at least one other vertex of γ. Therefore π is a branching vertex of the component of $\Gamma(G)$, containing γ. This component of

$\Gamma(G)$ is a connected Coxeter graph, so it cannot have other branching vertices. If γ contains more than one principal vertex then one of the vertices of γ must be branching in γ. But π is not branching in γ, and it is the only branching vertex of the component of $\Gamma(G)$, containing γ. Therefore the statement is true in this situation as well. $\qquad \square$

If a non-acceptable component γ does not contain principal vertices then we call all vertices of γ **regular**.

Let a non-acceptable component γ contain a principal vertex π and a branching vertex ρ of $\Gamma(G)$. Consider the vertices along the simple path in γ connecting π to ρ (including both). Let us call these vertices irregular, all other vertices of γ are called **regular**. Note that if ρ is an end vertex of γ (this is exactly the case (2) in the proof of Lemma 3.12) then all vertices of γ are irregular.

Corollary 3.13. *The regular vertices of non-acceptable components are the only admissible non-principal vertices in these components.*

Combining the statements of this subsection, we arrive to the following result:

Theorem 3.14. *Let G be a Coxeter group, naturally acting on V. Fix $x \in V$ such that $\operatorname{supp}_G x$ intersects every component of $\Gamma(G)$. The following is a complete list of admissible vertices and their multiplicities:*

(i)   every interior vertex from $\operatorname{supp}_G x$; its multiplicity equals 1,

(ii)   every boundary vertex from $\operatorname{supp}_G x$, except of those adjacent to at least one acceptable component of $\Gamma(G) \setminus \operatorname{supp}_G x$; its multiplicity equals 1,

(iii)   every principal vertex π of a component γ of $\Gamma(G) \setminus \operatorname{supp}_G x$; its multiplicity equals $1 + (\operatorname{card} G(\gamma)/\operatorname{card} G(\gamma \setminus \{\pi\}))$;

(iv)   every regular vertex π of a non-acceptable component γ; its multiplicity equals $1 + (\operatorname{card} G(\gamma)/\operatorname{card} G(\gamma \setminus \{\pi\}))$.

3.5. **Simplicial orbihedra.** A version of the next result is known (and is important in construction of special toroidal varieties) — it is due to Klyachko and Voskresenskii (Theorem 4 in [17]). They formulate it as a criterion of simpliciality of a cone obtained from a Weyl chamber by the action of a stabilizer group. We obtain this result as a direct corollary of Theorem 3.14.

A full dimensional G-orbihedron is called **simplicial** if there are exactly $\dim V$ faces of maximal dimension adjacent to every vertex of this polyhedron. One can easily see that if G is irreducible and $\operatorname{supp}_G x = \operatorname{ver}(G)$ then $\operatorname{Co}_G x$ is simplicial. It is not hard to present examples of non-simplicial G-orbihedra. A natural question is:

for which $x \in V$ the related G-orbihedron is simplicial?

A Coxeter graph is said to be of A_n **type** if it has no branching vertices and has no multiple edges.

Corollary 3.15. *Let G be an irreducible Coxeter group. Then $\mathrm{Co}_G\, x$ is simplicial if and only if the following is true:*

(i)   the graph $\Gamma(G) \setminus \mathrm{supp}_G\, x$ is of A_n type,

(ii)   every component of $\Gamma(G) \setminus \mathrm{supp}_G\, x$ contains an end vertex of $\Gamma(G)$ (a principal vertex),

(iii)   there are no vertices of $\mathrm{supp}_G\, x$ adjacent to more than one component of $\Gamma(G) \setminus \mathrm{supp}_G\, x$.

Proof. Due to explanations preceding Lemma 3.6 we only need to find out when the sum of multiplicities of all admissible vertices is exactly $\dim V = \mathrm{card\ ver}\,(G)$.

First, all vertices from $\mathrm{supp}_G\, x$ except of those adjacent to acceptable components, are admissible and have multiplicities 1. To each non-admissible vertex of $\mathrm{supp}_G\, x$ we assign an acceptable component γ adjacent to this vertex, and distinct non-admissible vertices of $\mathrm{supp}_G\, x$ get distinct acceptable components, due to the definitions. It may happen that there remains an acceptable component not assigned to any non-admissible vertex of $\mathrm{supp}_G\, x$. Every principal vertex of an acceptable component γ is admissible, with multiplicity at least $1 + \mathrm{card\ ver}\,(\gamma)$. The multiplicity is exactly $1 + \mathrm{card\ ver}\,(\gamma)$ if and only if γ is of A_n type, due to Lemma 3.5. So, the sum of multiplicities of admissible vertices in an acceptable component is greater or equal to the number of vertices in this component plus one = the number of vertices adjacent to this component (we agree that vertices belonging to the component are also adjacent to it). Therefore the sum of multiplicities of admissible vertices in all acceptable components is greater or equal to the number of vertices in these components plus the number of non-admissible vertices in $\mathrm{supp}_G\, x$, with equality if and only if all acceptable components are of A_n type, and none of non-admissible vertices from $\mathrm{supp}_G\, x$ is adjacent to more than one acceptable component.

Every non-acceptable component γ of $\Gamma(G) \setminus \mathrm{supp}_G\, x$ has admissible vertices each of multiplicity at least $1 + \mathrm{card\ ver}\,(\gamma) > \mathrm{card\ ver}\,(\gamma)$. So, the sum of multiplicities of admissible vertices in all non-acceptable components is strictly greater than the number of vertices in these components.

Therefore the sum of multiplicities of all admissible vertices equals to the overall number of vertices if and only if all components of $\Gamma(G) \setminus \mathrm{supp}_G\, x$ are acceptable, all are of A_n type, and none of vertices from $\mathrm{supp}_G\, x$ is adjacent to more than one component of $\Gamma(G) \setminus \mathrm{supp}_G\, x$. $\square$

4. Coxeter Orbihedra: faces of lower dimensions

Now we generalize Theorem 3.3 to obtain a complete description of all faces (not necessarily of codimension 1) of a G-orbihedron $\operatorname{Co}_G x$.

Each face ϕ of U of codimension 2 is a codimension 1 face in a face Φ of U of codimension 1, and, further, each face of U of codimension k is a codimension 1 face in a face of U of codimension $k-1$.

Theorem 3.3 provides a description of faces of $\operatorname{Co}_G x$ of codimension 1 in $V^{[x]}$. A face Φ of $\operatorname{Co}_G x$ of codimension 1 in $V^{[x]}$, containing x, is nothing else but $\operatorname{Co}_{\operatorname{Stab}_G \omega} x$, where $\omega \in \mathcal{W}_{G[x]}$ is such that $\operatorname{supp}_G x$ intersects every component of $\Gamma(G[x]) \setminus \{\pi(\omega)\}$. The face Φ is a full dimension convex subset of an affine hyperplane in $V^{[x]}$ orthogonal to ω. The faces of Φ are codimension 2 (in $V^{[x]}$) faces of $\operatorname{Co}_G x$. Let us project Φ onto the codimension 1 subspace $(\omega)^\perp$ in $V^{[x]}$. Then $\operatorname{proj}_\omega \Phi = \operatorname{Co}_{G_\omega} \operatorname{proj}_\omega x$. So, we are in the situation of a Coxeter group G_ω, and we can describe a face ψ of $\operatorname{proj}_\omega \Phi$, containing $\operatorname{proj}_\omega x$, with the help of Theorem 3.3:

$$\psi = \operatorname{Co}_{\operatorname{Stab}_{G_\omega} \kappa} \operatorname{proj}_\omega x,$$

where $\operatorname{supp}_{G_\omega} \kappa$ consists of one vertex of $\Gamma(G[x]_\omega)$ and $\operatorname{supp}_{G_\omega} \operatorname{proj}_\omega x$ intersects every component of $\Gamma(G[x]_\omega) \setminus \operatorname{supp}_{G_\omega} \kappa$, and $\operatorname{proj}_\omega x$ and κ are in one Weyl chamber of $G[x]_\omega$.

Keep in mind that $\Gamma(G[x]_z) = \Gamma(G[x]) \setminus \operatorname{supp}_G z$, $\operatorname{supp}_{G[x]_z} \operatorname{proj}_z x = \operatorname{supp}_{G[x]} x \setminus \operatorname{supp}_G z$, and $\kappa = \operatorname{proj}_\omega \tau$, $\tau \in \mathcal{W}_{G[x]}$ (Lemmas 2.6, 2.7, Corollary 2.8). Then $\operatorname{Stab}_{G_\omega} \kappa = \operatorname{Stab}_G \{\omega, \tau\}$. Repeating the same argument, we arrive to the following result:

Theorem 4.1. *Let G be a Coxeter group naturally acting on V. For every codimension k face ϕ of the G-orbihedron $\operatorname{Co}_G x$ there exists a unique set $\Omega = \Omega(\phi) \subset \mathcal{W}_G$, $\operatorname{card} \Omega = k$, of fundamental weights, belonging to the same Weyl chamber C, such that*
(i) $\operatorname{supp}_G x$ intersects every component of $\Gamma(G[x]) \setminus \operatorname{supp}_G \Omega$,
(ii) $\phi = \operatorname{Co}_{\operatorname{Stab}_G \Omega} x^*(C, G)$.
Moreover, for every set $\Omega \subset \mathcal{W}_G \cap C$, satisfying (i), the set ϕ defined in (ii) is a codimension $\operatorname{card} \Omega$ face of $\operatorname{Co}_G x$.

The set of all faces of a convex polyhedron is naturally partially ordered by the inclusion relation. We say that two convex polyhedra in V are **facially isomorphic** if their sets of faces are isomorphic as partially ordered sets. Obviously, such an isomorphism must preserve the dimensions of the faces and the number of vertices on the corresponding faces.

Corollary 4.2. *Let G be a Coxeter group, naturally acting in V. Two G-orbihedra $\operatorname{Co}_G x$ and $\operatorname{Co}_G y$ are facially isomorphic if and only if*

there exists an automorphism of the graph $\Gamma(G)$ transforming $\operatorname{supp}_G x$ into $\operatorname{supp}_G y$.

5. Coxeter-invariant Convex Polyhedra

Let G be a Coxeter group naturally acting on V. Consider a general convex G-invariant polyhedron U. Then the set $\operatorname{Extr} U$ of extreme points of U is also G-invariant, and therefore it is fibered into G-orbits. Let $I_G(U)$ denote the set of G-orbits in $\operatorname{Extr} U$. Then

$$U = \operatorname{conv} \bigcup_{i \in I_G(U)} i = \operatorname{conv} \bigcup_{\operatorname{Orb}_G x \in I_G(U)} \operatorname{Co}_G x.$$

Let

$$N_G(U) = \operatorname{card} I_G(U).$$

This number is a measure of complexity of the polyhedron U, in particular, if $N_G(U) = 1$, then U is a G-orbihedron. Let us refer to a G-invariant convex polyhedron U such that $N_G(U) = n$, as to a (G, n)-**polytope**. So, a G-orbihedron will also be called a $(G, 1)$-polytope.

Recall that

$$\operatorname{Carr}_G U = \operatorname{supp}_G \operatorname{Extr} U = \bigcup_{i \in I_G(U)} \operatorname{supp}_G i.$$

It is not hard to see that a polyhedron U is of full dimension if and only if the set $\operatorname{Carr}_G U$ intersects every component of $\Gamma(G)$. As before we may assume that the polyhedron U is of full dimension. If not, we switch to the subspace $\operatorname{span} U$ and to the group $G[U] = G|_{\operatorname{span} U}$. Again, $\Gamma(G[U])$ consists of those components of $\Gamma(G)$ which intersect $\operatorname{Carr}_G U$.

Obviously,

$$U^\circ = \bigcap_{i \in I_G(U)} i^\circ = \bigcap_{i \in I_G(U)} (\operatorname{conv} i)^\circ,$$

and if we are looking for the extreme points of U° we have to determine the extreme points of this intersection. The extreme points of each of the sets $(\operatorname{conv} i)^\circ$ are already described, they all are weights, i.e., their supports consist of one vertex. Fix a Weyl chamber C. Since the set U° is G-invariant, it is sufficient to find only extreme points of U° that are in C. Obviously, $C \cap \operatorname{Extr}(U^\circ) \subset \operatorname{Extr}(C \cap U^\circ)$, but these sets may be different. Note that for every i the set $C \cap (\operatorname{conv} i)^\circ$ is in fact the simplex

$$S_C(i) = \{\sum_j \lambda_j \omega_j : \lambda_j \geq 0, \sum_j \lambda_j m_G(i, \omega_j) \leq 1\}.$$

Here $\omega_j, j = 1, 2, \cdots, \dim V$, are the fundamental weights belonging to C. Obviously, the origin is a vertex of $S_C(i)$, let us call this vertex a trivial vertex. The non-trivial vertices of $S_C(i)$ are the vectors $\omega_j / m_G(i, \omega_j), j = 1, 2, \cdots, \dim V$. This simplex $S_C(i)$ is cut off the Weyl chamber C by the affine hyperplane

$$\Pi_C(i) = \{\sum_j \lambda_j \omega_j : \sum_j \lambda_j m_G(i, \omega_j) = 1\}.$$

Theorem 5.1. *Let G be a Coxeter group naturally acting in V, and let U be a (G, n)-polytope.*

(i)   If $y \in \operatorname{Extr}(U^\circ)$ then $\operatorname{supp}_G y$ consists of no more than n vertices;

(ii)   There cannot exist two distinct vectors in $C \cap \operatorname{Extr}(U^\circ)$ with coinciding supports consisting of exactly n vertices;

(iii)   If $y \in \operatorname{Extr}(U^\circ)$ then $\operatorname{Carr}_G U$ intersects every component of $\Gamma(G) \setminus \operatorname{supp}_G y$.

Proof. Fix a Weyl chamber C and assume that $y \in C$. The point $y \in C$ is an extreme point of U°, therefore it has to be an extreme point of $C \cap (U^\circ)$, so it is the intersection of $\dim V$ linearly independent boundary hyperplanes of $C \cap (U^\circ)$. All boundary hyperplanes of this set are either the walls of C or the hyperplanes $\Pi_C(i)$, $i = 1, 2, \ldots, n$. Therefore y belongs to no more than n affine hyperplanes $\Pi_C(i)$, hence it belongs to no less than $\dim V - n$ walls of C. So, y does not belong to at most n walls, which proves (i).

To prove (ii), it suffices to note that if y has a support of n vertices then it belongs to exactly $\dim V - n$ walls of C. Therefore, it must belong to all of n affine hyperplanes $\Pi_C(i)$. Since y is an extreme vector, these $\dim V$ hyperplanes have only one common point. Therefore, any other vector from $\operatorname{Extr}(C \cap U^\circ)$, having the same support, must coincide with this one.

To prove (iii), note that if $x_i, i = 1, 2, \ldots, n$, are representatives of n pairwise distinct G-orbits constituting the set of extreme vectors of our (G, n)-polytope, then the set $\{gx_i : g \in G, i = 1, 2, \ldots, n : \langle y, gx_i \rangle = m_G(y, x_i) = 1\}$ must span the whole space V. Therefore, by Lemma 2.9 for each i all vectors gx_i on the face must belong to the same $\operatorname{Stab}_G y$-orbit. So the orthogonal projections of these orbits to V_y must span the whole space V_y. Therefore $\bigcup_i \operatorname{supp}_{G_y} \operatorname{proj}_y x_i$ must intersect every component of $\Gamma(G_y) = \Gamma(G) \setminus \operatorname{supp}_G y$. Recalling Lemma 2.7, we see that

$$\bigcup_i \operatorname{supp}_{G_y} \operatorname{proj}_y x_i = \bigcup_i (\operatorname{supp}_G x_i \setminus \operatorname{supp}_G y) = \operatorname{Carr}_G U \setminus \operatorname{supp}_G y,$$

so the proof is completed.   □

The above result can be strengthened and complemented with a description of the codimension 1 face of U, associated with a given extreme vector of U°. For a convex G-invariant polyhedron U and for $y \in V$ let

$$m_G(U, y) = \max_{j \in I_G(U)} m_G(j, y),$$

$$I_G(U, y) = \{i \in I_G(U) : m_G(i, y) = m_G(U, y)\},$$

$$\operatorname{Carr}_G(U, y) = \bigcup_{i \in I_G(U, y)} \operatorname{supp}_G i.$$

As before, V^y denotes the intersection of all mirrors containing y. Recall that $U \cap V^y = \operatorname{proj}^y U$.

Theorem 5.2. *Let G be a Coxeter group naturally acting in V. For every codimension 1 face Φ of a (G, n)-polytope U there exists a unique subset $\gamma \subset \operatorname{ver}(G[U])$ and a unique vector y, $\operatorname{supp}_G y = \gamma$, such that*
  (i)   *card $\gamma \le n$,*
  (ii)   *y is a normal to a codimension 1 face of the polyhedron $U \cap V^y$ in the subspace V^y,*
  (iii) *$\operatorname{Carr}_G(U, y)$ intersects every component of $\Gamma(G[U]) \setminus \gamma$,*
  (iv)   *let $x_i \in i$ be such that $\langle x_i, y \rangle = m_G(i, y)$. Then*

$$\Phi = \operatorname{conv} \bigcup_{i \in I_G(U, y)} \operatorname{Co}_{\operatorname{Stab}_G y} x_i.$$

So, Φ is a $(\operatorname{Stab}_G y, k)$-polytope, where $k \le \operatorname{card} I_G(U, y)$.
  Moreover, for every $\gamma \subset \operatorname{ver}(G[U])$, $y \in V$, $\operatorname{supp}_G y = \gamma$, satisfying (i) — (iii), the set Φ defined in (iv), is a codimension 1 face of U.

Note that if $n = 1$ then $\operatorname{Carr}_G U = \operatorname{Carr}_G(U, y)$ for any y, so (iii) is formulated in terms of $\operatorname{Carr}_G U$ and γ only, and there always exists a unique y satisfying (ii). So, in this case Theorem 5.2 reduces to Theorem 3.3. This means that (i) and (iii) deliver a full description of $\operatorname{Extr}(U^\circ)$ in terms of $\operatorname{Carr}_G U$ and γ for $n = N_G(U) = 1$. Such a description is not possible for $n \ge 2$. We present some counterexamples in Theorem 5.10 below. Note that if $N_G(U) \ge \dim V$, then condition (i) in Theorem 5.1 is satisfied by any vector in V. We show that in this case any vector from V can serve as a vector from $\operatorname{Extr}(U^\circ)$ for a $(G, \dim V)$-polytope U, see Theorem 5.10 below.

Let us now describe the elements of $\operatorname{Extr}(U^\circ)$ having the minimal and maximal supports.

First, we describe all one-vertex supported elements of $\operatorname{Extr}(U^\circ)$ for a G-invariant convex polytope U.

Corollary 5.3. *Let U be a G-invariant convex polytope. Let π be a vertex of $\Gamma(G[U])$. The vector $\omega / m_G(U, \omega)$, $\pi(\omega) = \pi$, belongs to*

Extr (U°) *if and only if the set* $\mathrm{Carr}_G(U, \omega)$ *intersects every component of* $\Gamma(G[U]) \setminus \{\pi\}$.

Now let us describe the vectors from Extr (U°) whose supports consist of the maximal possible number of vertices, namely, of $N_G(U)$ vertices.

Corollary 5.4. *Let* U *be a* (G, n)*-polytope,* $n \le \dim V$. *Choose* $\gamma \subset$ ver $(G[U])$ *such that* $\mathrm{card}\, \gamma = n$. *There exists a unique* G*-orbit* $i \in I_G(U)$, $\mathrm{supp}_G\, i = \gamma$ *if and only if*
  (i) *the linear system*

$$\sum_{j \in \gamma} \lambda_j m_G(i, \omega_j) = 1, \quad i = 1, 2, \ldots, n, \quad i \in I_G(U),$$

has a unique solution, and all entries of this solution are positive,
  (ii) $\mathrm{supp}_G\, U$ *intersects every component of* $\Gamma(G[U]) \setminus \gamma$.

5.1. $(G, 2)$**-polytopes.** Let U be a $(G, 2)$-polytope. Then all vectors from Extr (U°) have supports consisting of one or two vertices of $\Gamma(G)$. It is easy to describe these supports in rather explicit geometric terms. Let $i_\pm$ denote the two G-orbits constituting $I_G(U)$. For $\pi \in$ ver $(G[U])$ choose $\omega \in \mathcal{W}_G$ such that $\pi(\omega) = \pi$, and let

$$\mu(\pi) = \frac{m_G(i_+, \omega)}{m_G(i_-, \omega)}.$$

Since $\pi \in$ ver $(G[U])$, it cannot happen that the numerator and the denominator of the fraction defining $\mu(\pi)$ are both zero, so $0 \le \mu(\pi) \le \infty$. Note that $\mu(\pi)$ does not depend upon the choice of ω, $\pi(\omega) = \pi$.

Consider the **canonical partition** of the set ver $(G[U])$ into the following three subsets:

$$I_+ = \{\pi \in \text{ver}\,(G[U]) : \mu(\pi) > 1\},$$

$$I_- = \{\pi \in \text{ver}\,(G[U]) : \mu(\pi) < 1\},$$

$$I_0 = \{\pi \in \text{ver}\,(G[U]) : \mu(\pi) = 1\}.$$

Considering the geometry of the related lines, one can easily verify that a linear system

$$\lambda_0 a_{00} + \lambda_1 a_{01} = 1$$

$$\lambda_0 a_{10} + \lambda_1 a_{11} = 1$$

with positive coefficients $a_{ij}, 0 \le i, j, \le 1$, has a unique solution with positive entries if and only if

$$\left(\frac{a_{01}}{a_{11}} - 1\right)\left(\frac{a_{00}}{a_{10}} - 1\right) < 0.$$

The above considerations lead to the following result:

Corollary 5.5. *Let U be a $(G, 2)$-polytope. Consider the canonical partition I_+, I_-, I_0 of the set $\mathrm{ver}\,(G[U])$, associated with the two G-orbits $i_\pm$ constituting $I_G(U)$.*

Choose a vertex $\pi \in \Gamma(G[U])$. There exists a vector $z \in \mathrm{Extr}\,(U^\circ)$ such that $\mathrm{supp}_G\, z = \{\pi\}$ if and only if **one of the following** *conditions is satisfied:*

(i) if $\pi \in I_\pm$, then $\mathrm{supp}_G\, i_\pm$ intersects every component of $\Gamma(G[U]) \backslash \{\pi\}$,

(ii) if $\pi \in I_0$, then $\mathrm{Carr}_G\, U = \mathrm{supp}_G\, i_+ \cup \mathrm{supp}_G\, i_-$ intersects every component of $\Gamma(G[U]) \setminus \{\pi\}$,

Choose two distinct vertices π, κ in $\Gamma(G[U])$. There exists a vector $z \in \mathrm{Extr}\,(U^\circ)$ such that $\mathrm{supp}_G\, z = \{\pi, \kappa\}$ if and only if **both of the following** *conditions are satisfied:*

(iii) $\mathrm{Carr}_G\, U$ intersects every component of $\Gamma(G[U]) \setminus \{\pi, \kappa\}$,

(iv) one of the vertices π, κ belongs to the set I_+, the other belongs to the set I_-.

If (iii - iv) hold then $z = g(\lambda_0 \omega + \lambda_1 \rho)$, where $g \in G$, and ω, ρ are fundamental weights (belonging to the same Weyl chamber) such that $\mathrm{supp}_G\, \omega = \{\pi\}, \mathrm{supp}_G\, \rho = \{\kappa\}$. Here (λ_0, λ_1) is the unique (positive) solution of the linear system

$$\lambda_0 m_G(i_+, \omega) + \lambda_1 m_G(i_+, \rho) = 1$$

$$\lambda_0 m_G(i_-, \omega) + \lambda_1 m_G(i_-, \rho) = 1.$$

The above vectors z form an exhaustive list of elements of $\mathrm{Extr}\,(U^\circ)$.

5.2. A_n-invariant polytopes and spectra of Hermitian matrices.

It has been known for quite a long time that the geometry of A_n-orbihedra is very important for many natural problems related to the spectral theory of Hermitian operators. Recently there was a breakthrough, due mostly to A.A. Klyachko, in an old problem of description of the possible spectra of sums of Hermitian matrices with given spectra (see [9, 12]). Here we present some simple remarks related to such problems.

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{R}^n$, let $\mathrm{diag}\,\alpha$ denote the diagonal matrix having α as its diagonal. So, diag is a real linear mapping from $\mathbf{R}^n$ to the real linear space of Hermitian $n \times n$ matrices. Let

$$\mathrm{Orb}_U\, \alpha = \{u(\mathrm{diag}\,\alpha)u^* : u \in U(n)\}$$

be the set of Hermitian matrices, whose spectrum is $\{\alpha_1, \cdots, \alpha_n\}$. Here $U(n)$ denotes the unitary group acting on $\mathbf{C}^n$. For any Hermitian $n \times n$ matrix A let $\mathrm{Diag}\, A$ denote the diagonal of A, viewed as a vector from $\mathbf{R}^n$. So, Diag is a real linear mapping from the real linear space of Hermitian $n \times n$ matrices to $\mathbf{R}^n$. Certainly, $\mathrm{Diag}\,(\mathrm{diag}\,\alpha) = \alpha$. To simplify the formulations we restrict ourselves to Hermitian matrices

with zero trace, the space of such $n \times n$ matrices is denoted by $H_0(n)$. Note that $iH_0(n)$ is the Lie algebra of the Lie group $U(n)$, and the action of $U(n)$ on $H_0(n)$ is adjoint action of a Lie group on its Lie algebra. Also, let V_{n-1} denote the $(n-1)$-dimensional subspace of $\mathbf{R}^n$ consisting of vectors with the zero sum of coordinates. Obviously, if $\alpha \in V_{n-1}$ then $\mathrm{Orb}_U \mathrm{diag}\,(\alpha) \subset H_0(n)$. For a matrix $A \in H_0(n)$ let $\mathrm{Spec}\,A = \{\lambda_1, \cdots, \lambda_n\} \subset \mathbf{R}$ denote its spectrum, let $\mathrm{spec}\,A$ denote the set of vectors $(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)}) \in V_{n-1}$, where σ runs over all permutations of $\{1, 2, \cdots, n\}$. So, $\mathrm{spec}\,A$ is an A_{n-1}-orbit. We treat spec as a mapping from $H_0(n)$ to the space of A_{n-1}-orbits. The following result, due to I. Schur and A. Horn (see [10]), establishes a beautiful connection between A_{n-1}-orbihedra and spectra of Hermitian matrices with zero trace:

Theorem 5.6. *Let $\alpha \in V_{n-1}$. Then*

$$\mathrm{Diag}\,(\mathrm{Orb}_U \mathrm{diag}\,(\alpha)) = \mathrm{Co}_{A_{n-1}} \alpha.$$

In other words, for any $A \in H_0(n)$ $\mathrm{Diag}\,\mathrm{Orb}_U A$ is a convex polyhedron and $\mathrm{spec}\,A$ is the set of its extreme vectors:

$$\mathrm{Extr}\,(\mathrm{Diag}\,\mathrm{Orb}_U A) = \mathrm{spec}\,A.$$

Let $\alpha, \beta \in V_{n-1}$. An important problem going back to H. Weyl is to compute the set

$$\mathrm{spec}\,(\mathrm{Orb}_U \mathrm{diag}\,\alpha + \mathrm{Orb}_U \mathrm{diag}\,\beta).$$

After important contributions by H. Weyl, Ky Fan, V.B. Lidskii, H. Wielandt, A. Horn, and others, this problem was recently solved by A. Klyachko [12]. The ideas of this solution came from Representation Theory and Algebraic Geometry. It should be noted that the connections of this problem with representation theory of Lie groups were known for at least 50 years.

We present two simple results, using the ideas of the preceding sections.

Theorem 5.7. *Let $\alpha, \beta \in V_{n-1}$. Then*

$$\mathrm{spec}\,(\mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\alpha + \mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\beta) = \mathrm{Co}_{A_{n-1}}(\alpha^* + \beta^*).$$

Proof. The set $\mathrm{spec}\,(\mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\alpha + \mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\beta)$ is convex – one can verify this by a straightforward computation. Using the definitions and the Schur-Horn Theorem, we obtain

$$\mathrm{spec}\,(\mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\alpha + \mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\beta)$$

$$\subset \mathrm{Diag}\,(\mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\alpha + \mathrm{conv}\,\mathrm{Orb}_U \mathrm{diag}\,\beta)$$

$$= \mathrm{conv}\,(\mathrm{Diag}\,(\mathrm{Orb}_U \mathrm{diag}\,\alpha) + \mathrm{Diag}\,(\mathrm{Orb}_U \mathrm{diag}\,\beta))$$

$$= \mathrm{conv}\,(\mathrm{Co}_{A_{n-1}} \alpha + \mathrm{Co}_{A_{n-1}} \beta)$$

$$= \mathrm{Co}_{A_{n-1}} \alpha + \mathrm{Co}_{A_{n-1}} \beta = \mathrm{Co}_{A_{n-1}}(\alpha^* + \beta^*).$$

To prove the last equality, we compute the polar sets of both convex sets and verify that they coincide:

$$(\text{Co}_{A_{n-1}} \alpha + \text{Co}_{A_{n-1}} \beta)^\circ = \{z \in V_{n-1} : \forall g, h \in A_{n-1} \ \langle z, g\alpha + h\beta \rangle \leq 1\}$$

$$= \{z \in V_{n-1} : \langle z^*, \alpha^* \rangle + \langle z^*, \beta^* \rangle \leq 1\}$$

$$= \{z \in V_{n-1} : \forall g \in A_{n-1} \ \langle z, g(\alpha^* + \beta^*) \rangle \leq 1\}$$

$$= (\text{Co}_{A_{n-1}}(\alpha^* + \beta^*))^\circ.$$

On the other hand, vectors $g(\alpha^* + \beta^*)$, $g \in A_{n-1}$, (which are the extreme vectors of $\text{Co}_{A_{n-1}}(\alpha^* + \beta^*)$) obviously belong to $\text{spec}(\text{Orb}_U \text{diag } \alpha + \text{Orb}_U \text{diag } \beta)$. So, since we know that

$$\text{spec}(\text{conv Orb}_U \text{diag } \alpha + \text{conv Orb}_U \text{diag } \beta)$$

is convex then the assertion follows. □

Let us reformulate the result choosing the following Weyl chamber $C = \{x_1 \geq x_2 \geq \cdots \geq x_n\} \subset V_{n-1}$ and letting $x^* = x^*(A_{n-1}, C)$. Note that the related fundamental weights are the orthogonal projections of the vectors $(1, 1, \cdots, 1, 0, \cdots, 0)$ (k ones, $1 \leq k \leq n - 1$) onto V_{n-1}.

Corollary 5.8. *Let $\alpha, \beta \in V_{n-1}$. Then*

$$\gamma \in \text{spec}(\text{conv Orb}_U \text{diag } \alpha + \text{conv Orb}_U \text{diag } \beta)$$

if and only if

$$\forall k, \ k \leq n - 1, \ \sum_{i=1}^{k} (\gamma^*)_i \leq \sum_{i=1}^{k} (\alpha^*)_i + \sum_{i=1}^{k} (\beta^*)_i.$$

Essentially repeating the considerations in the proof of the previous Theorem, we arrive to the following result:

Theorem 5.9. *Let $\alpha, \beta \in V_{n-1}$. Then*

$$\text{spec}(\text{conv}(\text{Orb}_U \text{diag } \alpha \bigcup \text{Orb}_U \text{diag } \beta))$$

is the $(A_{n-1}, 2)$-polytope $\text{conv}(\text{Orb}_{A_{n-1}} \alpha \bigcup \text{Orb}_{A_{n-1}} \beta)$.

Using the description of the extreme vectors of polars of $(G, 2)$-polytopes we can describe the vectors from

$$\text{spec}(\text{conv}(\text{Orb}_U \text{diag } \alpha \bigcup \text{Orb}_U \text{diag } \beta)$$

in terms of a system of linear inequalities.

5.3. $(G, 2)$-polytopes – some counterexamples.

Theorem 5.10. *Let G be an irreducible Coxeter group.*

(a) There exist $(G, 2)$-polytopes such that all elements from $\mathrm{Extr}\,(U^\circ)$ have the same one-vertex support, provided $\dim V \geq 3$.

(b) If $\Gamma(G)$ is not branching and $\dim V \geq 4$, then there exist two distinct $(G, 2)$-polytopes U_1, U_2 such that $\mathrm{Carr}_G U_1 = \mathrm{Carr}_G U_2$, all elements of $\mathrm{Extr}\,(U_1^\circ)$ have the same one-vertex support π, all elements of $\mathrm{Extr}\,(U_2^\circ)$ have the same one-vertex support κ, but $\pi \neq \kappa$.

(c) For every $z \in V$ such that $\mathrm{supp}_G z = \mathrm{ver}\,(G)$ there exists a $(G, \dim V)$-polytope U such that $z \in \mathrm{Extr}\,(U^\circ)$.

Proof. (a) If $\Gamma(G)$ is not branching, then let ω be a non-extremal fundamental weight (i.e., the related vertex $\pi = \pi(\omega)$ is not one of the end vertices π_1, π_2 of the Coxeter graph $\Gamma(G)$). Then

$$\mathrm{Extr}\,(Co_G\omega)^\circ = \mathrm{Orb}_G \frac{\omega_1}{m_G(\omega, \omega_1)} \bigcup \mathrm{Orb}_G \frac{\omega_2}{m_G(\omega, \omega_2)}$$

where $\pi(\omega_i) = \pi_i$, $i = 1, 2$, – this immediately follows from Theorem 3.2. Then $U = (\mathrm{Co}_G \omega)^\circ$ is a $(G, 2)$-polytope, $\mathrm{Carr}_G U = \{\pi_1, \pi_2\}$, and

$$\mathrm{Extr}\,(U^\circ) = \mathrm{Extr}\,(\mathrm{Co}_G \omega) = \mathrm{Orb}_G \omega_1.$$

So, all elements of $\mathrm{Extr}\,(U^\circ)$ have the same one-vertex support $\pi(\omega)$. If $\Gamma(G)$ is branching, choose π to be one of the three end vertices of $\Gamma(G)$.

(b) In the previous construction choose two distinct fundamental weights ω, τ, associated to non-end vertices of the Coxeter graph. It is possible since $\mathrm{card}\,\mathrm{ver}\,(G) = \dim V \geq 4$.

(c) Put $U = (\mathrm{Co}_G z)^\circ$. Obviously, $\mathrm{Extr}\,(U^\circ) = \mathrm{Extr}\,(\mathrm{Co}_G z) = \mathrm{Orb}_G z$. On the other hand, according to Theorem 3.2,

$$\mathrm{Extr}\,U = \{\omega/m_G(\omega, z) : \mathrm{supp}_G z \text{ intersects every component of}$$

$$\Gamma(G) \setminus \{\pi(\omega)\}\} = \bigcup_{\pi(\omega) \in \mathrm{ver}\,(G)} \mathrm{Orb}_G \omega/m_G(\omega, z)$$

so $N_G(U) = \mathrm{card}\,\mathrm{ver}\,(G) = \dim V$. $\qquad\square$

6. ORBIHEDRA FOR QUASI-COXETER GROUPS

6.1. Quasi-Coxeter groups. Let us start with an important example:

Definition 6.1. *Let $V = M_{n,m}(\mathbf{R})$, the set of matrices having n columns of m real entries each. Define $B_{n,m}$ as the group of operators acting on V by permuting the columns, permuting elements in each individual column, and performing sign changes on any number of entries.*

Groups $B_{n,m}$ naturally arise as symmetry groups for the so called **mixed norms**: choose a B_n-invariant norm $l : \mathbf{R}^n \to \mathbf{R}_+$, and a B_m-invariant norm $L : \mathbf{R}^m \to \mathbf{R}_+$, treat an element $A \in M_{n,m}(\mathbf{R})$ as a string $(a_1, a_2, \cdots, a_n)$ of n vectors from $\mathbf{R}^m$, then define the mixed norm (lL) as follows:

$$(lL)(A) = l(L(a_1), L(a_2), \cdots, L(a_n)).$$

The unit balls of such norms are important examples of $B_{n,m}$-invariant convex bodies.

Notably, $B_{n,m}$ is not generated by reflections across hyperplanes, but it does have a close relationship to the Coxeter group B_m. A reformulation of the above definition makes this relationship more evident. Specifically, $B_{n,m} = S_n(B_m)$ where $S_n(G)$ has the following definition. Let S_n denote the group of permutations of the set $\{1, 2, \cdots, n\}$.

Definition 6.2. *Let G be a Coxeter group naturally acting on V. Define*

$$S_n(G) = S_n \times G^n = \{\sigma \times (\prod_{i=1}^{n} g_i) : \sigma \in S_n, g_i \in G, \ 1 \le i \le n\}.$$

Group $S_n(G)$ acts on V^n as follows:

$$(\sigma \times (\prod_{i=1}^{n} g_i))(v_1, v_2, \ldots, v_n) = (g_1 v_{\sigma(1)}, g_2 v_{\sigma(2)}, \ldots, g_n v_{\sigma(n)}).$$

In other terms, the action of $S_n(G)$ on V^n is induced by the natural action of G on V.

Again, group $S_n(G)$ is not generated by reflections across hyperplanes, but it has a Coxeter subgroup G^n.

Now consider a more general situation: let K be a finite group of operators acting on a real finite dimensional space V. Consider all reflections across hyperplanes contained in K. Let $G = G(K)$ denote the subgroup generated by all these reflections. Assume that G acts effectively on V. Then G is called a **Coxeter subgroup** of K.

Let $G\backslash K = \{Gk = \{gk : g \in G\} : k \in K\}$ be the left homogeneous space. For every $x \in V$ we have

$$\mathrm{Orb}_K x = \bigcup_{Gk \in G\backslash K} \mathrm{Orb}_G kx.$$

Therefore

$$\mathrm{Co}_K x = \mathrm{conv} \bigcup_{Gk \in G\backslash K} \mathrm{Co}_G kx.$$

So, every K-orbihedron is a convex G-invariant polyhedron, and $I_G(\mathrm{Co}_K x) \le \mathrm{card}\,(G\backslash K)$. The number $\mathrm{card}\,(G\backslash K) = (K : G)$ is called the index of the subgroup G in the group K.

Consider a finite group K of operators acting on a real finite dimensional space V containing a Coxeter subgroup (= an effectively acting subgroup generated by reflections across hyperplanes) of index smaller that $\dim V$. We call such group K a **quasi-Coxeter** group.

The techniques presented in the previous sections allows to study the convex structure of K-orbihedra for quasi-Coxeter groups K.

6.2. $S_2(G)$-orbihedra.

Consider the case $K = S_2(G)$, where G is an irreducible Coxeter group naturally acting on V (irreducibility is actually not very important, but this assumption simplifies some formulations). This group contains an index 2 Coxeter subgroup G^2, naturally acting on V^2 :

$$(g_0, g_1)(v_+, v_-) = (g_0 v_+, g_1 v_-).$$

There is an obvious action of S_2 on V^2 by permutations of the components. Let

$$\sigma(v_+, v_-) = (v_-, v_+).$$

We call this operator σ a **flip**. Let $v = (v_+, v_-) \in V^2$. Consider a $S_2(G)$-orbihedron (which is also a $(G^2, 2)$-polytope)

$$U = \operatorname{Co}_{S_2(G)} v = \operatorname{conv}\left(\operatorname{Co}_{G^2} v \bigcup \operatorname{Co}_{G^2} \sigma(v)\right).$$

Corollary 5.5 provides a complete description of the set $\operatorname{Extr}(U^\circ)$, but we can simplify this description by incorporating flip symmetries of $\Gamma(G^2)$ (to be defined in the next paragraph) and U.

The Coxeter graph $\Gamma(G^2)$ is the disjoint union of two copies Γ_+, Γ_- of the connected graph $\Gamma(G)$:

$$\Gamma(G^2) = \Gamma_+ \bigsqcup \Gamma_-.$$

There is a natural automorphism of $\Gamma(G^2)$, interchanging Γ_+ and Γ_-. Slightly abusing notation, we call this automorphism a **flip**. So, the graph $\Gamma(G^2)$ is also flip-invariant. The carrier set $\operatorname{Carr}_{G^2} U$ is also flip-invariant. Therefore, it intersects both components of $\Gamma(G^2)$, so $G^2[U] = G^2$.

Let C be a Weyl chamber for G, then $C^2 = C \times C$ is a Weyl chamber for G^2. Let $\omega_j \in C \cap \mathcal{W}_G$, $1 \leq j \leq \dim V$, be the fundamental weights. Let $\pi_j, 1 \leq j \leq \dim V$, denote the related vertices of $\Gamma(G)$. Then $\omega_j^+ = (\omega_j, 0)$, $\omega_j^- = (0, \omega_j)$, $1 \leq j \leq \dim V$, are the fundamental weights of G^2, belonging to C^2. Let π_j^i, $1 \leq j \leq \dim V$, denote the related vertices of Γ_i, $i = \pm$, so we have $\pi(\omega_j^i) = \pi_j^i$. We wish to describe the extreme vectors of U° in terms of group G rather than in terms of group G^2.

Let

$$\nu_j = \frac{m_G(v_+, \omega_j)}{m_G(v_-, \omega_j)}, \ 1 \leq j \leq \dim V.$$

Since $\Gamma(G)$ is connected then the numerator and the denominator of this fraction can both vanish if and only if $v_+ = v_- = 0$, which is obviously excluded. Consider the canonical partition of $\mathrm{ver}\,(G)$ associated with vectors v_+ and v_- :

$$J_+ = \{\pi_j \in \mathrm{ver}\,(G) : \nu_j > 1\},$$
$$J_- = \{\pi_j \in \mathrm{ver}\,(G) : \nu_j < 1\},$$
$$J_0 = \{\pi_j \in \mathrm{ver}\,(G) : \nu_j = 1\}.$$

Let $(J_k)^i = \{\pi_j^i \in \mathrm{ver}\,(G^2) : \pi_j \in J_k\}$, $i = \pm$, $k = 0, \pm$.

As before,

$$\mu(\pi_j^i) = \frac{m_{G^2}(v, \omega_j^i)}{m_{G^2}(\sigma(v), \omega_j^i)}, \quad i = \pm, \ 1 \leq j \leq \dim V.$$

Obviously,

$$m_{G^2}(v, \omega_j^i) = m_G(v_i, \omega_j), \ \ m_{G^2}(\sigma(v), \omega_j^i) = m_G(v_{-i}, \omega_j), \ \ i = \pm, \ 1 \leq j \leq \dim V.$$

Therefore the following is true for the canonical partition of $\mathrm{ver}\,(G^2)$, associated with vectors v and $\sigma(v)$:

$$I_+ = \{\pi_j^+ \in \mathrm{ver}\,(G^2) : \nu_j > 1\} \bigcup \{\pi_j^- \in \mathrm{ver}\,(G^2) : \nu_j < 1\}$$

$$= (J_+)^+ \bigcup (J_-)^-,$$

$$I_- = \{\pi_j^+ \in \mathrm{ver}\,(G^2) : \nu_j < 1\} \bigcup \{\pi_j^- \in \mathrm{ver}\,(G^2) : \nu_j > 1\}$$

$$= (J_+)^- \bigcup (J_-)^+,$$

$$I_0 = \{\pi_j^i \in \mathrm{ver}\,(G^2) : \nu_j = 1, \ i = \pm\} = (J_0)^+ \bigcup (J_0)^-.$$

The flip maps the sets I_+ and I_- one onto another and leaves the set I_0 invariant.

Applying Corollary 5.5 we arrive to the following result:

Theorem 6.3. *Let $U = \mathrm{Co}_{S_2(G)}\, v$, where G is an irreducible Coxeter group, naturally acting on space V, and $v = (v_+, v_-) \in V^2$. The set $\mathrm{Extr}\,(U^\circ)$ can be described as follows:*

1. Take $\pi_j \in J_s$, $s = \pm$. There exists $z \in \mathrm{Extr}\,(U^\circ)$, $\mathrm{supp}_{G^2}\, z = \pi_j^i$, $i = +$ or $-$, if and only if the following is true:

(i) $\mathrm{supp}_G\, v_s$ intersects every component of $\Gamma(G) \backslash \{\pi_j\}$, and $v_{-s} \neq 0$. In this case $z = g\omega_j^i / m_G(v_s, \omega_j)$, $g \in G$.

2. Take $\pi_j \in J_0$. There exists $z \in \mathrm{Extr}\,(U^\circ)$, $\mathrm{supp}_{G^2}\, z = \pi_j^i$, $i = +$ or $-$, if and only if the following is true:

(ii) $\mathrm{supp}_G\, v_+ \cup \mathrm{supp}_G\, v_-$ intersects every component of $\Gamma(G) \backslash \{\pi_j\}$. In this case $z = g\omega_j^i / m_G(v_0, \omega_j)$, $g \in G$.

3. Take $\pi_j, \pi_k \in \mathrm{ver}\,(G)$, $j \neq k$. There exists $z \in \mathrm{Extr}\,(U^\circ)$, $\mathrm{supp}_{G^2}\, z = \{\pi_j^i, \pi_k^i\}$, $i = +$ or $-$, if and only if the following is true:

(iii) $\operatorname{supp}_G v_+ \cup \operatorname{supp}_G v_-$ *intersects every component of* $\Gamma(G) \setminus \{\pi_j, \pi_k\}$.

(iv) *one of the vertices* π_j, π_k *belongs to* J_+, *the other - to* J_-.

In this case $z = g(\lambda_0 \omega_j^i + \lambda_1 \omega_k^i)$, $g \in G$, $i = \pm$, (λ_0, λ_1) *is the unique (positive) solution of the linear system*

$$\lambda_0 m_G(v_+, \omega_j) + \lambda_1 m_G(v_+, \omega_k) = 1$$

$$\lambda_0 m_G(v_-, \omega_j) + \lambda_1 m_G(v_-, \omega_k) = 1.$$

4. *Take* $\pi_j, \pi_k \in \operatorname{ver}(G)$ *(the case* $j = k$ *is not excluded here). There exists* $z \in \operatorname{Extr}(U^\circ)$, $\operatorname{supp}_{G^2} z = \{\pi_j^i, \pi_k^{-i}\}$, $i = +$ *or* $-$, *if and only if the following is true:*

(v) $\operatorname{supp}_G v_+ \cup \operatorname{supp}_G v_-$ *intersects every component of of the graphs* $\Gamma(G) \setminus \{\pi_j\}$ *and* $\Gamma(G) \setminus \{\pi_k\}$,

(vi) *both of the vertices* π_j, π_k *belong to* J_i, $i = +$ *or* $-$.

In this case $z = g(\lambda_0 \omega_j^i + \lambda_1 \omega_k^{-i})$, $g \in G$, $i = \pm$, (λ_0, λ_1) *is the unique (positive) solution of the linear system*

$$\lambda_0 m_G(v_+, \omega_j) + \lambda_1 m_G(v_-, \omega_k) = 1$$

$$\lambda_0 m_G(v_-, \omega_j) + \lambda_1 m_G(v_+, \omega_k) = 1.$$

The above is an exhaustive list of elements of $\operatorname{Extr}(\operatorname{Co}_{S_2(G)} v)^\circ$.

A less explicit form of this result for $G = B_2$ was obtained in [21] by a hard (non-computer) computation, based on an algorithm calculating the extreme rays of a polyhedral cone defined by a system of linear inequalities (this algorithm is known as the Chernikova's algorithm, or the Double Description Method).

These results have an application to the Operator Interpolation Theory in the spirit of [19, 18, 20], which we discuss in the next sections.

7. OPERATOR INTERPOLATION

Our initial interest in the convex geometry of orbihedra was motivated by an approach to operator interpolation developed by the last two authors, for a complete exposition see [18, 19]. Let us briefly describe the main features of this approach.

Let $G \subset O(V)$ be a subgroup of orthogonal operators on a real finite dimensional Euclidean space V. We wish to describe the **envelope** of G (denoted env (G)) — the set of linear operators on V transforming every G-invariant convex closed set into itself:

env $(G) = \{T \in \operatorname{End} V : TU \subset U$ for every closed convex G-invariant

$$U \subset V\}.$$

Obviously, env (G) is a convex closed semigroup of linear operators, containing the convex hull of the group G.

A collection $\{U_\alpha, \alpha \in A\}$ of G-invariant convex closed sets is called G-**sufficient** if

$$T \in \mathrm{env}\,(G) \iff \forall \alpha \in A \quad TU_\alpha \subset U_\alpha.$$

We would like to describe some natural G-sufficient collections. A collection consisting of G-orbihedra is called a **simple collection**. A collection consisting of polar sets of G-orbihedra is called a **dual simple collection**.

Example (Calderon–Mityagin Theorem). Let G be the Coxeter group B_n. It acts on $\mathbf{R}^n$ as follows:

$$(x_1, x_2, \cdots, x_n) \mapsto (s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, \cdots, s_n x_{\sigma(n)}),$$

where σ is a permutation of $\{1, 2, \cdots, n\}$, and $s_k = \pm 1$. Let

$$U_1 = \{(x_k) \in \mathbf{R}^n : \sum_k |x_k| \le 1\},$$

$$U_\infty = \{(x_k) \in \mathbf{R}^n : \max_k |x_k| \le 1\}.$$

A finite dimensional version of the celebrated Calderon-Mityagin interpolation theorem (see [7, 16]) asserts that if a linear operator $T : V \to V$ is such that $TU_1 \subset U_1$ and $TU_\infty \subset U_\infty$ then $TU \subset U$ for every closed convex B_n-invariant $U \subset V$ (in other words, the contraction property of T with respect to U_1 and U_∞ can be interpolated to all closed convex B_n-invariant bodies). In our terms this means that the collection U_1, U_∞ is a (both simple and dual simple) B_n-sufficient collection. It was shown in [19] that this collection is actually the smallest B_n-sufficient collection, more precisely, it is a subset of the Hausdorff closure of any B_n-sufficient collection (up to scaling).

There is a natural duality between the spaces $\mathrm{End}\,V$ of linear operators in V and the tensor product space $V \otimes V$:

$$(T, \sum x_i \otimes y_i) = \sum \langle T y_i, x_i \rangle.$$

Therefore there is a natural notion of the polar set, in particular,

$$G^\circ = \{S \in V \otimes V : \forall g \in G (g, S) \le 1\}.$$

Sufficient collections can be described in terms of the following sets in $V \otimes V$:

$$\mathcal{A}_G = \{x \otimes y \in V \otimes V : m_G(x, y) \le 1\} = G^\circ \bigcap \{\text{rank 1 tensors}\},$$

$$K_G = \mathrm{conv}\,\mathcal{A}_G.$$

Since $\mathcal{A}_G$ is obviously closed and $V \otimes V$ is finite dimensional, then K_G is closed. One can show (see [19]) that K_G is bounded if and only if G

acts irreducibly, which we assume henceforth. Let $\text{Extr } K_G$ denote the set of extreme elements of K_G. One can show that

$$\text{Extr } K_G \subset \mathcal{A}_G.$$

Since K_G is a compact convex set in a finite dimensional space then, by the Krein–Milman Theorem and the Caratheodory Theorem,

$$K_G = \text{conv Extr } K_G.$$

Note that K_G is invariant with respect to the following **tensor flip**: $x \otimes y \rightarrow y \otimes x$. Therefore the set $\text{Extr } K_G$ is also flip-invariant.

For every set $U \subset V$ let

$$S(U) = \{x \otimes y \in V \otimes V : x \in U, y \in U^\circ\}.$$

Obviously, if U is convex and G-invariant then $S(U) \subset \mathcal{A}_G$. For any $x \otimes y \in \mathcal{A}_G$ (i.e., such that $m_G(x, y) \leq 1$) we have $x \otimes y \in S(\text{Co}_G x)$ and $x \otimes y \in S((\text{Co}_G y)^\circ)$.

It is not hard to see that a collection $(U_\alpha : \alpha \in A)$ of G-invariant convex closed sets is G-sufficient if and only if

$$\text{Extr } K_G \subset \overline{\bigcup_{\alpha \in A} S(U_\alpha)}.$$

This observation leads to the following constructions: let

$$\mathcal{N}_G = \{x \in V : \exists y \in V, x \otimes y \in \text{Extr } K_G\},$$

and let us consider the following **simple canonical collection**:

$$\mathcal{C}_G = \{\text{Co}_G x : x \in \mathcal{N}_G\},$$

and the following **dual simple canonical collection**:

$$\mathcal{C}_G^\circ = \{(\text{Co}_G x)^\circ : x \in \mathcal{N}_G\}.$$

It is easy to see that each of the canonical collections is G-sufficient, and each of them is minimal in some natural sense — see [19, 18].

7.1. **Non-canonical sufficient collections.** It is often very difficult to compute the sets $\text{Extr } K_G$ and $\mathcal{N}_G$. Therefore it is interesting to find larger sets and construct larger G-sufficient collections.

The set

$$\mathcal{K}_G = \{x \otimes y \in V \otimes V : x \in \text{Extr}\,(\text{Co}_G y)^\circ, y \in \text{Extr}\,(\text{Co}_G x)^\circ\}$$

is a very natural set of this type, it is contained in $\mathcal{A}_G$ and contains $\text{Extr } K_G$ (in fact, we have no examples of groups with $\text{Extr } K_G \neq \mathcal{K}_G$, though believe that such examples do exist). Let

$$\widehat{\mathcal{N}}_G = \{x \in V : \exists y \in V, \ x \otimes y \in \mathcal{K}_G\},$$

and we arrive to the **quasi-canonical** G-sufficient collections

$$\widehat{\mathcal{C}}_G = \{\text{Co}_G x : x \in \widehat{\mathcal{N}}_G\}$$

and
$$\widehat{\mathcal{C}}_G^\circ = \{(\mathrm{Co}_G x)^\circ : x \in \widehat{\mathcal{N}}_G\}.$$

Let us construct several other G-sufficient collections. Let $\mathcal{N}_0(G) = V$, and let

$$\mathcal{N}_{s+1}(G) = \bigcup\{\mathrm{Extr}\,(\mathrm{Co}_G z)^\circ :\ z \in \bigcup_{w \in \mathcal{N}_s(G)} \mathrm{Extr}\,(\mathrm{Co}_G w)^\circ\}.$$

It is not hard to show that

$$V = \mathcal{N}_0(G) \supset \mathcal{N}_1(G) \supset \mathcal{N}_2(G) \supset \cdots \supset \widehat{\mathcal{N}}_G \supset \mathcal{N}_G.$$

Let

$$\mathcal{C}_s(G) = \{\mathrm{Co}_G x : x \in \mathcal{N}_s(G)\}.$$

Obviously,

$$\mathcal{C}_1(G) \supset \mathcal{C}_2(G) \supset \cdots \supset \widehat{\mathcal{C}}_G \supset \mathcal{C}_G.$$

Therefore, all these collections are G-sufficient (but not minimal, if different from $\mathcal{C}_G$). The actual construction of these collection heavily depends upon the knowledge of the convex structure of G-orbihedra. This was our initial motivation for the study of these problems.

As it was shown in [19], the equality $\mathcal{C}_1(G) = \mathcal{C}_G$ is equivalent to the fact that "interpolation in the canonical collection is described by the real method", see [19] for details. Such assertions are important in Operator Interpolation. It actually means that there exist very simple decompositions of elements of $\mathcal{A}_G$ into convex combinations of elements of Extr K_G, and all convex G-invariant bodies can be obtained from the bodies of the canonical collection by rather simple constructions (by the so called real method). In particular, in this case one may interpolate not only linear operators but also many non-linear ones.

7.2. **Canonical collections for Coxeter groups.** If G is an irreducible Coxeter group then the set Extr K_G was explicitly computed in terms of weights (see [19]):

$$\mathrm{Extr}\ K_G$$

$$= \{\frac{\omega \otimes \tau}{m_G(\omega, \tau)} : \omega, \tau \in \mathcal{W}_G, \pi(\omega), \pi(\tau) \text{ are distinct end vertices of } \Gamma(G)\}.$$

Elements of Extr K_G are called **Birkhoff's tensors** — see [6, 15] for an explanation how the Birkhoff's tensors are related to the Birkhoff's description ([4]) of the extreme points of the set of doubly stochastic matrices. So, for every irreducible Coxeter group the set Extr $(\mathrm{env}\ G)^\circ$ is explicitly computed. In fact conv $G = \mathrm{env}\ G$ if the Coxeter graph $\Gamma(G)$ is not branching (this is proven in [6, 15] for all irreducible Coxeter groups with non-branching graphs with the only exception of the group H_4 for which this assertion is still a conjecture). As for the case when $\Gamma(G)$ is branching, it was shown in [15] that conv $G \neq \mathrm{env}\ G$.

These results were recently applied in [13] to a description of linear isomorphisms of the convex hulls of Coxeter groups.

Thus in the case of an irreducible Coxeter group we have

$$\mathcal{N}_G = \{\omega \in \mathcal{W}_G : \pi(\omega) \text{ is an end vertex of } \Gamma(G)\}.$$

Also, one can show that $\mathcal{C}_1 = \mathcal{C}_G$ for all irreducible Coxeter groups, so the interpolation here is "described by the real method" — see [7] for $G = B_n$ (even in the infinite dimensional setting), and [19] for all other irreducible Coxeter groups. Moreover, in this case the canonical collections have some additional nice extremal properties — see [18, 19].

In the case of $G = B_n$ both canonical collections coincide with the collection U_1, U_∞ described above.

7.3. **Sufficient collections for non-Coxeter groups.** All above notions and constructions may be generalized to the case when G is a bounded semigroup of operators on V — see [20]. In particular, we may consider the semigroup of operators contracting every mixed norm on $V = \mathbf{R}^{n \times m}$. It is possible to compute the canonical collections for this semigroup, see [18]. As it was mentioned before, the mixed norms are $B_{n,m}$-invariant. Not all $B_{n,m}$-invariant norms are mixed norms — one can present counterexamples. The group $B_{n,m}$ is not a Coxeter group, and our initial goal was to construct canonical (or, at least, quasi-canonical) collections for this group. A calculation of the quasi-canonical collections for $G = B_{2,2}$ was carried out in [21]. It was based on the calculation of the extreme vectors of the polar sets of $B_{2,2}$-orbihedra. Since we now have a rather detailed description of these vectors for groups $S_2(G)$, we can calculate quasi-canonical and even canonical collections for these groups. This calculation is rather lengthy and we plan to discuss it in a separate publication.

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E-mail address: `namccart@mit.edu, david.ogilvie@midway.uchicago.edu, zobin@math.wm.edu, veronica@math.wm.edu`

Department of Mathematics, College of William and Mary, Williamsburg, VA 23187