# WHITNEY'S PROBLEM ON EXTENDABILITY OF FUNCTIONS AND AN INTRINSIC METRIC

Nahum Zobin

Abstract. We consider the space of functions with bounded $(k+1)$-th derivatives in a general domain in $\mathbb{R}^n$. Is every such function extendible to a function of the same class defined on the whole $\mathbb{R}^n$? H.Whitney showed that the equivalence of the intrinsic (=geodesic) metric in this domain to the Euclidean one is sufficient for such extendability. There was an old conjecture (going back to H.Whitney) that this equivalence is also necessary for extendability. We disprove this conjecture and construct examples of domains in $\mathbb{R}^2$ such that the above extendability holds but the analogous property for smaller $k$ fails. Our study is based on a duality approach.

## Introduction

Let $\Omega$ be a bounded connected open set in $\mathbb{R}^n$. Consider the following Sobolev function space (see, e.g., [14, ch.V, §6.2] ):

$$W^{k+1}_\infty(\Omega) = \{f \in C^k(\Omega) : \forall \alpha \in \mathbb{Z}^n_+, |\alpha| = k+1,\ f^{(\alpha)} \in L^\infty(\Omega)\}$$

Here, as usually, $|\alpha| = \sum_{i=1}^n \alpha^i$ for $\alpha = (\alpha^1, \dots, \alpha^n) \in \mathbb{Z}^n_+$, $f^{(\alpha)}$ denotes the corresponding (distributional) partial derivative, $C^k(\Omega)$ denotes the space of $k$ times continuously differentiable functions, $L^\infty(\Omega)$ denotes the space of essentially bounded functions on $\Omega$. Let $W^{k+1}_\infty(\mathbb{R}^n)|_\Omega$ denote the space of restrictions to $\Omega$ of functions from $W^{k+1}_\infty(\mathbb{R}^n)$. Obviously,

$$W^{k+1}_\infty(\mathbb{R}^n)|_\Omega \subset W^{k+1}_\infty(\Omega)$$

One can easily present examples of domains $\Omega$ for which the inclusion is proper. We are studying the following

**Problem.** *Under what conditions on* $\Omega$

$$(EP_{k,l}) \qquad W^{l+1}_\infty(\mathbb{R}^n)|_\Omega \supset W^{k+1}_\infty(\Omega)?$$

*In particular,*

$$(EP_{k,k}) \qquad W^{k+1}_\infty(\mathbb{R}^n)|_\Omega = W^{k+1}_\infty(\Omega)?$$

---

1991 *Mathematics Subject Classification.* 46E35.

*Key words and phrases.* Extension of smooth functions, Whitney theorem, intrinsic metric.

A large part of the research was made while the author was a member of the Afula Research Institute (University of Haifa, Israel). On this stage the research was partially supported by grants from the Ministry of Absorption, the Ministry of Science and Technology (Israel) and by the Rashi Foundation (France-Israel)

$$(EP_{k,0}) \qquad W^1_\infty(\mathbb{R}^n)|_\Omega \supset W^{k+1}_\infty(\Omega)?$$

If $\Omega$ satisfies $(EP_{k,l})$ we write: $\Omega \in (EP_{k,l})$. In his remarkable work [16] Hassler Whitney gave a description of functions from $W^{k+1}_\infty(\mathbb{R}^n)|_\Omega$ in terms of their behavior on $\Omega$ (see Theorem 6 below). As a corollary of this description, he formulated [15] a simple sufficient condition for $\Omega$ to yield $(EP_{k,k})$. Let us first introduce the following notation: For $x, y \in \Omega$ let

$$d_\Omega(x,y) = \text{ infimum of lengths of polygonal paths in } \Omega \text{ joining } x \text{ and } y$$

Obviously, $d_\Omega(x,y) \geq d_{\mathbb{R}^n}(x,y)$. $d_\Omega$ is called the **intrinsic** (or **geodesic**) **metric** in $\Omega$. Consider the following geometric condition on $\Omega$ :

$$\text{(W)} \qquad \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{d_\Omega(x,y)}{d_{\mathbb{R}^n}(x,y)} < \infty$$

The sets $\Omega$, satisfying (W), are called **Whitney-regular** (see, e.g., [8]).

**Proposition 1.** *(H. Whitney [15]) (W) implies* $(EP_{k,k})$, $k = 0, 1, \dots$ .

One can easily show that the converse holds for $k = 0$ (a stronger result is contained in [8]):

**Proposition 2.** $(EP_{0,0})$ *implies (W).*

*Proof.* Fix $x \in \Omega$, consider the function $f(y) = d_\Omega(x,y)$. Check that $f \in W^1_\infty(\Omega)$ and

$$\sup_{\substack{z\in\Omega \\ |\alpha|=1}} |f^{(\alpha)}(z)| = 1$$

If $\Omega \in (EP_{0,0})$, then $f$ is extendible to $\tilde{f} \in W^1_\infty(\mathbb{R}^n)$ and the Open Mapping Theorem guarantees that one can choose $\tilde{f}$ such that

$$\sup_{\substack{z\in\mathbb{R}^n \\ |\alpha|=1}} |\tilde{f}^{(\alpha)}(z)| \leq C \sup_{\substack{z\in\Omega \\ |\alpha|=1}} |f^{(\alpha)}(z)| = C.$$

So,

$$d_\Omega(x,y) = |f(y) - f(x)| = |\tilde{f}(y) - \tilde{f}(x)| \leq d_{\mathbb{R}^n}(x,y) \max_{\substack{z\in\mathbb{R}^n \\ |\alpha|=1}} |\tilde{f}^{(\alpha)}(z)| \leq C\, d_{\mathbb{R}^n}(x,y). \blacksquare$$

There was a long standing Conjecture (usually called the **Whitney Conjecture**):

for any $k = 1, 2, \dots,$ $(EP_{k,k})$ implies $(W)$.

It was shown that this is really true, provided $\Omega$ is a simply connected domain in $\mathbb{R}^2$. To my best knowledge, the first proof was found by myself and is contained in my PhD thesis (Voronezh State University, USSR, 1975), but it was never published. V.N. Konovalov in 1984 published an independent proof of this result ([11]). Recently I proved the following

**Theorem 3.** *If $\Omega$ is a finitely connected bounded domain in $\mathbb{R}^2$, then for any fixed $k = 0, 1, \dots,$ $(EP_{k,k})$ implies (W).*

The proof will be published in a separate paper.

The main result of the present paper is the following Theorem, showing, in particular, the failure of the Whitney Conjecture for $\mathbb{R}^n$, $n \geq 2$.

**Main Theorem.** *For any $k \geq 1$ there exists a bounded connected domain $\Omega \subset \mathbb{R}^2$ such that*

$$W^{k+1}_\infty(\Omega) = W^{k+1}_\infty(\mathbb{R}^2)|_\Omega$$

*but*

$$W^k_\infty(\Omega) \not\subset W^1_\infty(\mathbb{R}^2)|_\Omega.$$

In our notations the assertion of the Main Theorem is: for $n = 2$

$$(EP_{k,k}) \backslash (EP_{k-1,0}) \neq \emptyset.$$

In particular, the domain $\Omega$ does not belong to $(EP_{0,0})$, so (W) fails and $\Omega$ delivers a counterexample to the Whitney Conjecture. In fact, we show more, namely, we prove that

$$\bigcap_{l=1}^{k} W^l_\infty(\Omega) \not\subset L^\infty(\Omega).$$

Certainly, the Main Theorem implies that such domains exist in any greater dimension. For $n = 2$ this domain is necessarily infinitely connected, due to Theorem 3. For $n \geq 3$ this domain may be chosen to be a topological ball. We are dealing only with the Sobolev spaces $W^k_\infty(\Omega)$. Certainly, the problems of extension arise in other classes of functions, including general Sobolev spaces $W^k_p(\Omega)$, Lipschitz spaces, Zygmund spaces, nonquasianalytic classes (see, e.g., [1–4, 6, 7, 9, 10, 13, 14]). Each time there arises a problem of description of domains allowing extendability of functions with preservation of class. It is interesting to note that that there appear versions of the condition (W), proving to be necessary and (or) sufficient for various types of extendability. For example, consider the Sobolev spaces $W^k_p$. A.P. Calderón [4] showed that $W^k_p(\Omega) = W^k_p(\mathbb{R}^n)|_\Omega$ for any $k$ and $p$, $1 < p < \infty$, provided $\partial\Omega$ is Lipschitz. E.M. Stein extended this result to cover the cases $p = 1, \infty$ (see [14]). Finally, P.W. Jones [9] showed that this result holds for so-called $(\varepsilon - \delta)$-domains (this means, roughly speaking, that any two points may be connected by a not very long and not very thin tube in $\Omega$) and proved that this result is final: in the case of bounded finitely connected planar domains the $(\varepsilon - \delta)$-condition is also necessary for the equality

$$W^1_2(\Omega) = W^1_2(\mathbb{R}^2)|_\Omega$$

(for earlier results, concerning simply connected domains, see [6, 7]). V.G. Maz'ja (see [13]) gave an example of a simply connected planar domain not satisfying the $(\varepsilon - \delta)$-condition but such that

$$W^1_p(\Omega) = W^1_p(\mathbb{R}^2)|_\Omega \text{ for } p \neq 2.$$

I don't know whether the $(\varepsilon - \delta)$-condition is necessary for the equality

$$W^2(\Omega) = W^2(\mathbb{R}^2)|$$

As it is shown in the present paper for $p = \infty$, the geometric conditions of extendability for $k = 1$ and for greater $k$ do not coincide. Analogues of Whitney's theorem were proved for $L^p$-spaces (see [10]), for nonquasianalytic classes (see [1]). It would be very interesting to apply these results to obtain geometric conditions of extendability for these cases. The paper is organized as follows: in §1 we present a reduction of the problem, based on duality principles, in §2 we construct the domain $\Omega$, basing on postulated properties of special domains, called labyrinths, in §3 we present a construction of labyrinths – the main building blocks of the domain $\Omega \in (EP_{k,k}) \setminus (EP_{k-1,0})$. **Acknowledgments.** I am deeply grateful to Evsey Dyn'kin and Yoram Sagher for very stimulating and helpful discussions and remarks. I am greatly indebted to Charles Fefferman for important remarks. I appreciate very much numerous valuable discussions with Yu.A. Brudnyi, P. Kuchment, V. Lin and P. Shvartsman, in particular, I am grateful to Yu.A. Brudnyi for interesting comments and references. I am very thankful to Veronica Zobin for help and support.

## §1. Duality

It is well known (see, e.g., [14, ch.V, §6.2, ch.VI, §2.3]) that $W^{k+1}_\infty(\mathbb{R}^n)$ is the space of $C^k$-functions whose derivatives of order $k$ yield the Lipschitz condition. This is equally true for convex domains and therefore we obtain the following description of the space $W^{k+1}_\infty(\Omega)$ (see, e.g., [12, Ch.1]): For $x, y \in \Omega$ let

$$[xy] = \{(1-\lambda)x + \lambda y : \, 0 \le \lambda \le 1\}$$

be the straight line segment joining $x$ and $y$.

**Proposition 4.**

$$W^{k+1}_\infty(\Omega) = \{f \in C^k(\Omega) : \, \exists C, \; \forall x, y, \; [xy] \subset \Omega, \; \forall z \in \mathbb{R}^n,$$

$$|\sum_{|\alpha| \le k} f^{(\alpha)}(x) \frac{(z-x)^\alpha}{\alpha!} - f^{(\alpha)}(y) \frac{(z-y)^\alpha}{\alpha!}| \le C \int_{[xy]} |z-w|^k \, d|w|\}.$$

Let $f^{(p)}(x)$ denote the $p$-th Frechèt derivative of $f$ at the point $x \in \Omega$, i.e., $f^{(p)}(x)$ is a symmetric $p$-linear form on $\mathbb{R}^n$. So, $f^{(p)}$ belongs to the space $\mathcal{F}(\Omega, Sym_p\mathbb{R}^n)$ of functions on $\Omega$ with values in the space $Sym_p\mathbb{R}^n$ of symmetric $p$-linear forms on $\mathbb{R}^n$. The space $Sym_p\mathbb{R}^n$ is in a natural duality with the space $(\mathbb{R}^n)^{\odot p}$ of symmetric tensors. If $a \in \mathbb{R}^n$, then $a^{\otimes p} \in (\mathbb{R}^n)^{\odot p}$, and

$$< f^{(p)}(x), a^{\otimes p} > := f^{(p)}(x) \underbrace{(a, \dots, a)}_{p \text{ times}}$$

So, we may assert that

$$W^{k+1}_\infty(\Omega) = \{f \in C^k(\Omega) : \; \exists C, \; \forall\, x, y, \, [xy] \subset \Omega, \; \forall z \in \mathbb{R}^n,$$

$$|\sum^{k} < f^{(p)}(x), \frac{(z-x)^{\otimes p}}{p!} > - < f^{(p)}(y), \frac{(z-y)^{\otimes p}}{p!} > | \le C \int_{[xy]} |z-w|^k d|w|\}.$$

Consider the following (obviously injective) mapping:

$$j_k : W_\infty^{k+1}(\Omega) \to \mathcal{F}(\Omega, \bigoplus_{p=0}^{k} Sym_p \mathbb{R}^n)$$

$$(j_k f)(x) = (f(x), f'(x), \dots, f^{(k)}(x))$$

We want to describe the image $j_k(W_\infty^{k+1}(\Omega))$ (see, e.g., [12, Ch.1, 14, ch.VI, §2.3].)

**Proposition 5.**

$$j_k(W_\infty^{k+1}(\Omega)) =$$

$$= \{F = (f_0, f_1, \dots, f_k) \in \mathcal{F}(\Omega, \bigoplus_{p=0}^{k} Sym_p \mathbb{R}^n) : \exists\, C,\ \forall x, y,\ [xy] \subset \Omega,\ \forall z \in \mathbb{R}^n,$$

$$|\sum_{p=0}^{k} < f_p(x), \frac{(z-x)^{\otimes p}}{p!} > - < f_p(y), \frac{(z-y)^{\otimes p}}{p!} > | \le C \int_{[xy]} |z-w|^k d|w|\}$$

Now let us introduce the following linear space:

$$\mathcal{M}(\Omega, \prod_{p=0}^{k} (\mathbb{R}^n)^{\odot p}) = \{\sum_{i \in I} \mu_i \otimes \delta_{x_i} : \mu_i \in \prod_{p=0}^{k} (\mathbb{R}^n)^{\odot p},\ x_i \in \Omega, |I| < \infty\}$$

($\delta_x$ is the $\delta$-measure supported at the point $x$). We treat elements of $\mathcal{M}(\Omega, \prod_{p=0}^{k} (\mathbb{R}^n)^{\odot p})$ as finitely supported measures on $\Omega$ taking values in $\prod_{p=0}^{k} (\mathbb{R}^n)^{\odot p}$. The spaces $\mathcal{M}(\Omega, \prod_{p=1}^{k} (\mathbb{R}^n)^{\odot p})$ and $\mathcal{F}(\Omega, \bigoplus_{p=0}^{k} Sym_p \mathbb{R}^n)$ are in a natural duality

$$< F, \sum_{i \in I} \mu_i \otimes \delta_{x_i} > = \sum_{i \in I} < F(x_i), \mu_i > .$$

This is a nondegenerate bilinear form and one can easily see that $\mathcal{F}(\Omega, \bigoplus_{p=0}^{k} Sym_p(\mathbb{R}^n))$ is naturally identified with the space of all linear functionals on the space $\mathcal{M}(\Omega, \prod_{p=0}^{k} (\mathbb{R}^n)^{\odot p})$. Let $x \in \Omega,\ a \in \mathbb{R}^n$. Consider the following element:

$$m^k(a; x) = (1, a, \frac{a^{\otimes 2}}{2!}, \dots, \frac{a^{\otimes k}}{k!}) \otimes \delta_x \in \mathcal{M}(\Omega, \prod_{p=0}^{k} (\mathbb{R}^n)^{\odot p}).$$

Then we may write

$$j_k(W_\infty^{k+1}(\Omega)) = \{F \in \mathcal{F}(\Omega, \bigoplus_{p=0}^{k} Sym_p \mathbb{R}^n) : \exists C,\ \forall x, y,\ [xy] \subset \Omega,\ \forall z \in \mathbb{R}^n,$$

$$| < F, m^k(z-x; x) - m^k(z-y; y) > | \le C \int |z-w|^k d|w|\}.$$

Consider the following subspace of $\mathcal{M}(\Omega, \prod_{p=0}^{k}(\mathbb{R})^{\odot p})$ :

$$M_{k+1}^{\infty}(\Omega) = span\,\{m^k(z-x;x) - m^k(z-y;y) : x, y \in \Omega, z \in \mathbb{R}^n\}$$

For $\mu \in M_{k+1}^{\infty}(\Omega)$ put

$$\|\mu\|_{k+1}^{\Omega} = inf\{\sum |\lambda_i| \int_{[x_i y_i]} |z_i - w|^k d|w| :$$

$$\mu = \sum_i \lambda_i [m^k(z_i - x_i; x_i) - m^k(z_i - y_i; y_i)],\ [x_i y_i] \subset \Omega, z_i \in \mathbb{R}^n\}$$

Then we may write:

$$j_k(W_\infty^{k+1}(\Omega)) =$$

$$= \{F \in \mathcal{F}(\Omega, \bigoplus_{p=0}^{k} Sym_p \mathbb{R}^n) : \exists C,\ \forall \mu \in M_{k+1}^{\infty}(\Omega),\ | < F, \mu > | \leq C\|\mu\|_{k+1}^{\Omega}\}$$

This means that $j_k(W_\infty^{k+1}(\Omega))/[M_{k+1}^{\infty}(\Omega)]^{\perp}$ is naturally identified with the Banach dual space to the normed space $M_{k+1}^{\infty}(\Omega)$. One can easily prove that $j_k f \in [M_{k+1}^{\infty}(\Omega)]^{\perp}$ if and only if $f$ is a polynomial of degree $\leq k$. So $j_k$ establishes an isomorphism between the spaces $W_\infty^{k+1}(\Omega)/\mathcal{P}ol_{\leq k}$ and $[M_{k+1}^{\infty}(\Omega)]'$. It may be interesting to note that the space $\prod_{p=0}^{\infty}(\mathbb{R}^2)^{\odot p}$ is the Fock space – the quantum state space for a system of particles, $m^\infty(a;x) = exp^{\otimes} a \otimes \delta_x$, where $exp^{\otimes} a = \sum_{p=0}^{\infty} \frac{a^{\otimes p}}{p!}$ belongs to the Fock space. Now we present a version of the Whitney Theorem, describing the space $W_\infty^{k+1}(\mathbb{R}^n)|_\Omega$. The version is essentially due to G. Glaeser [5], see [12, Ch.1].

**Theorem 6.** *(H. Whitney)*

$$j_k(W_\infty^{k+1}(\mathbb{R}^n)|_\Omega) = \{F \in \mathcal{F}(\Omega, \bigoplus_{p=0}^{k} Sym_p \mathbb{R}^n) : \exists C,\ \forall x, y \in \Omega,\ \forall z \in \mathbb{R}^n,$$

$$| < F, m^k(z-x;x) - m^k(z-y;y) > | \leq C \int_{[xy]} |z-w|^k d|w|\}.$$

HERE WE DO NOT ASSUME, THAT $[xy] \subset \Omega$. Comparing $j_k(W_\infty^{k+1}(\Omega))$ and $j_k(W_\infty^{k+1}(\mathbb{R}^n)|_\Omega)$, we deduce the following

**Theorem 7.** $W_\infty^{k+1}(\Omega) = W_\infty^{k+1}(\mathbb{R}^n)|_\Omega$ *if and only if*

$$\exists C,\ \forall x, y \in \Omega,\ \forall z \in \mathbb{R}^n,$$

$$\|m^k(z-x;x) - m^k(z-y;y)\|_{k+1}^{\Omega} \leq C \int_{[xy]} |z-w|^k d|w|.$$

Comparing $j(W^k(\Omega))$ and $j(W^1(\mathbb{R}^n)|)$ we deduce the following

**Theorem 8.** *$W^k_\infty(\Omega) \subset W^1_\infty(\mathbb{R}^n)|_\Omega$ if and only if*

$$\exists C,\ \forall x, y \in \Omega,$$

$$\|m^{k-1}(y-x;x) - m^{k-1}(0;y)\|^\Omega_k \leq C|x-y|.$$

These theorems give geometric conditions on $\Omega$, necessary and sufficient for the belongings $\Omega \in (EP_{k,k})$ and $\Omega \in (EP_{k-1,0})$, respectively. So, wishing to construct $\Omega \in (EP_{k,k}) \backslash (EP_{k-1,0})$, we must ensure that

$$\exists C,\ \forall x, y \in \Omega,\ \forall z \in \mathbb{R}^n,$$

$$\|m^k(z-x;x) - m^k(z-y;y)\|^\Omega_{k+1} \leq C \int_{[xy]} |z-w|^k d|w|,$$

but there must exist $x_i, y_i \in \Omega$ such that

$$\|m^{k-1}(y_i - x_i; x_i) - m^{k-1}(0; y_i)\|^\Omega_k > i|x_i - y_i|.$$

In this paper we prefer to give an explicit construction of a function

$$\Psi \in \left(\bigcap_{l=1}^k W^l_\infty(\Omega)\right) \backslash L_\infty(\Omega)$$

instead of using Theorem 8.

## §2. Construction of a special domain $\Omega$

We are going to construct a special bounded domain $\Omega$ in $\mathbb{R}^2$. It will be constructed as a union of an increasing uniformly bounded sequence of domains $\Omega_n$ with real analytic boundaries $\partial\Omega_n$, $\bar{\Omega}_n \subset \Omega_{n+1}$, $n = 1, 2, \dots$. We put $\Omega_0 = \{a\}$. Choose a countable dense subset $Z \subset \mathbb{R}^2$, and represent it as a union of an increasing sequence of finite sets:

$$Z = \bigcup_{n=0}^\infty Z_n,\ \ Z_0 \subset Z_1 \subset Z_2 \subset \dots.$$

We shall inductively construct finite sets $T_0 \subset T_1 \subset T_2 \subset \dots$, such that $T_n$ will be a $\frac{1}{n}$-net in $\bar{\Omega}_n$. and therefore $\bigcup_{n=1}^\infty T_n$ will be dense in $\Omega$. Every $\bar{\Omega}_n$ will be equipped with a smooth function $\Psi_n$, such that

$$\Psi_n^{(\alpha)}(a) = 0 \text{ for } |\alpha| < k,$$

$$(\Omega_n, 1) \qquad \sup_{\substack{x \in \bar{\Omega}_n \\ |\alpha| = l}} |\Psi_n^{(\alpha)}(x)| \leq h_l\left(1 - \frac{1}{n+1}\right),\ 1 \leq l \leq k,\ h_k = 1.$$

These functions will be extensions of each other:

$$(\Omega_n, 2) \qquad \Psi_n|_{\bar{\Omega}_{n-1}} = \Psi_{n-1},\ n \geq 2$$

$$(\Omega_n, 3) \qquad \exists x_n \in \bar{\Omega}_n : \Psi_n(x_n) \geq n.$$

So the function $\Psi$ on $\Omega$, defined as

$$\Psi|_{\Omega_n} = \Psi_n,$$

will belong to $\bigcap_{l=1}^{k} W_\infty^l(\Omega)$, will yield the equalities $\Psi^{(\alpha)}(a) = 0$ ($|\alpha| < k$) but will be unbounded, so

$$\bigcap_{l=1}^{k} W_\infty^l(\Omega) \not\subset L_\infty(\Omega).$$

The sets $\bar{\Omega}_n$ will have the following property:

$$\text{for any } x, y \in T_{n-1} \text{ and for any } z \in Z_{n-1}$$

$$(\Omega_n, 4) \qquad \|m^k(z-x;x) - m^k(z-y;y)\|_{M^\infty_{k+1}(\Omega_n)} \leq C(k) \int_{[xy]} |z-w|^k d|w|.$$

So for any $x, y \in \bigcup_{k=0}^{\infty} T_k$ and for any $z \in Z$

$$\|m^k(z-x;x) - m^k(z-y;y)\|_{M^\infty_{k+1}(\Omega)} \leq C(k) \int_{[xy]} |z-w|^k d|w|.$$

and, recalling that $Z$ is dense in $\mathbb{R}^2$ and $\bigcup_{k=0}^{\infty} T_k$ is dense in $\Omega$, we see that this estimate holds for any $x, y \in \Omega$ and for any $z \in \mathbb{R}^2$. Keeping in mind Theorem 7, we easily conclude that

$$W_\infty^{k+1}(\mathbb{R}^2)|_\Omega = W_\infty^{k+1}(\Omega).$$

The main building blocks for our inductive construction of $\Omega_n$'s are special sets, called labyrinths. Given two points $a, b \in \mathbb{R}^2$, a vector $z \in \mathbb{R}^2$ and a number $N$, we construct a labyrinth $\mathcal{L}_{a,b,N}(z)$. (i) $\mathcal{L}_{a,b,N}(z)$ is the closure of a domain with a real analytic boundary, (ii) $\mathcal{L}_{a,b,N}(z) \subset \left\{ t \in \mathbb{R}^2 : |t - \frac{a+b}{2}| \leq \frac{|a-b|}{2} \right\}$, (iii) $a, b \in \partial\mathcal{L}_{a,b,N}(z)$ (iv) $\|m^k(z-a;a) - m^k(z-b;b)\|_{M^\infty_{k+1}(\mathcal{L}_{a,b,N}(z))} \leq C(k) \int_{[ab]} |z-w|^k d|w|$ (v) there exists a function $\Psi_{a,b,z,N} \in C^\infty(\mathcal{L}_{a,b,N}(z))$ such that: (*) $\Psi_{a,b,z,N}$ is identically zero in a neighborhood of $a$; (**) $\Psi_{a,b,z,N}$ is constant in a neighborhood of $b$; $\Psi_{a,b,z,N}(b) = N$. (***) $\sup_{\substack{x \in \mathcal{L}_{a,b,N}(z) \\ |\alpha| = l}} |\Psi^{(\alpha)}_{a,b,z,N}(x)| \leq h_l$, $1 \leq l \leq k$, $h_k = 1$.

Assuming that we are able to construct the labyrinths $\mathcal{L}_{a,b,N}(z)$ and the functions $\Psi_{a,b,z,N}$, we are presenting an inductive procedure of constructing $\Omega_n$'s. **Step 1.** We may assume that the set $Z_0$ consists of one vector $z_0$. Fix an open circle $C_0$ of radius 1, centered at $a$. All $\Omega_n$ will be subsets of $C_0$. Take an arbitrary point $b \in C_0$, $|a-b| = \frac{1}{2}$, and let $\Omega_1 = \mathcal{L}_{a,b,2}(z_0)$. Take $T_0 = \{a, b\}$, $\Psi_1 = \frac{1}{2}\Psi_{a,b,z_0,2}$, $x_1 = b$. The properties $(\Omega_1, 1-4)$ are easily verified. **Step $n$.** Now assume that $\Omega_n$ is already constructed together with the function $\Psi_n$ and the set $T_{n-1} \subset \bar{\Omega}_{n-1} \subset \Omega_n$. Choose a finite $\frac{1}{n}$-net $S_n \subset \bar{\Omega}_n$ and let

$$T_n = T_{n-1} \bigcup S_n$$

Consider the set
$$\omega_{n+1} = \bar{\Omega}_n \bigcup ( \bigcup_{x_i, x_j \in T_n} [x_i x_j])$$

Then consider the function $\Psi_n$ which is assumed to be already constructed on $\bar{\Omega}_n$. $\Psi_n \in C^\infty(\bar{\Omega}_n)$, $\partial\Omega_n$ is real analytic, so $\Psi_n$ may be extended to a $C^\infty$-function $\tilde{\Psi}_n$ on $\mathbb{R}^2$. Since
$$\sup_{\substack{x \in \bar{\Omega}_n \\ |\alpha| = l}} |\Psi_n^{(\alpha)}(x)| \le h_l(1 - \frac{1}{n}),\ 1 \le l \le k,$$

there exists a neighborhood $\tilde{\Omega}_n \supset \bar{\Omega}_n$ (we may assume that $\partial\tilde{\Omega}_n$ is real analytic) such that
$$\sup_{\substack{x \in \tilde{\Omega}_n \\ |\alpha| = l}} |\tilde{\Psi}_n^{(\alpha)}(x)| \le h_l \left( 1 - \frac{1}{n} + \frac{1}{4} \left( \frac{1}{n} - \frac{1}{n+1} \right) \right),\ \ 1 \le l \le k.$$

Consider $(\omega_{n+1} \cap \tilde{\Omega}_n) \backslash \Omega_n$. Since $\partial\Omega_n$ and $\partial\tilde{\Omega}$ are real analytic, this set consists of a finite number of straight line segments, subdivided into subsegments by intersection points. Consider only those subsegments, one end of which belongs to $\partial\Omega_n$ (if both ends belong to $\partial\Omega_n$, we split this subsegment into two by the middle point and consider the resulting two subsegments). Choose a point in each of these subsegments and surround it by a small disk so that these disks are pairwise disjoint and are contained in $\tilde{\Omega}_n \backslash \bar{\Omega}_n$. Let us erase the parts of the subsegments contained inside the constructed disks. We replace them by the constructions shown at Fig.1.

Fig. 1

The number of "teeth" is equal to $|Z_n|$. The ends of the "teeth", which belong to the connected component of $\Omega_n$, are called $a_p$ $(p = 1, 2, \dots, |Z_n|)$, the remaining ends are called $b_p$ $(p = 1, 2, \dots, |Z_n|)$ – $a_p$ and $b_p$ are corresponding to each other. Consider the labyrinths
$$\mathcal{L}_{a_p, b_p, A_n}(z_p)\ (z_p \in Z_n)$$

and insert them between $a_p$ and $b_p$ $(p = 1, 2, \dots, |Z_n|)$. (Certainly, we assume that the disks whose diameters are the segments $[a_p b_p]$ are pairwise disjoint.) The numbers $A_n$ are chosen as follows: let $M_n = \sup\limits_{x\in\tilde{\Omega}_n} |\tilde{\Psi}_n(x)|$, then

$$A_n = M_n \cdot l(n) \left[\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right]^{-1},$$

where $l(n)$ will be presented further. Put

$$\breve{\omega}_{n+1} = (\omega_{n+1} \backslash (\text{erased disks})) \bigcup (\text{ inserted constructions}) \bigcup (\text{ inserted labyrinths}).$$

Now consider the functions $\frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})\Psi_{a_p,b_p,z_p,A_n}$ on $\mathcal{L}_{a_p,b_p,A_n}(z_p)$ and extend them to the inserted constructions by zero beyond $a_p$, and by the constant

$$\frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})A_n = M_n \cdot l(n)$$

beyond $b_p$. Then these functions are further extended to $\bar{\Omega}_n$ by zero. We get the function $\phi_0$, defined on $\breve{\omega}_{n+1}$. There is another function – $\tilde{\Psi}_n(x)$ – defined on $\tilde{\Omega}_n$; $\tilde{\Omega}_n$ contains the set $\breve{\omega}_{n+1}$ where $\phi_0$ is defined. Consider the function $\mu_n = \tilde{\Psi}_n + \phi_0$. It is well defined on $\breve{\omega}_{n+1}$, it coincides with $\Psi_n$ on $\bar{\Omega}_n$,

$$\sup_{\substack{x\in\breve{\omega}_{n+1}\\|\alpha|=l}} |\mu_n^{(\alpha)}(x)| = \sup_{\substack{x\in\breve{\omega}_{n+1}\\|\alpha|=l}} |(\tilde{\Psi}_n + \phi_0)^{(\alpha)}(x)| \leq$$

$$\leq h_l\left[\frac{1}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right) + 1 - \frac{1}{n}\right] + \frac{h_l}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = h_l\left(1 - \frac{1}{n} + \frac{3}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right).$$

Let us estimate $\mu_n$ beyond the points $b_i$ :

$$\mu_n(x) = \tilde{\Psi}_n(x) + \phi_0(x) \geq \phi_0(x) - M_n =$$

$$= l(n) \cdot M_n - M_n = (l(n) - 1)M_n \geq (l(n) - 1)n.$$

We consider the function

$$\Psi_{n+1}(x) = \chi_n(\mu_n(x))$$

where $\chi_n$ is constructed as follows:

$$\chi_n'(t) = \begin{cases} 1, & t \leq max(n+1, M_n) \\ 0, & t \geq M_n(l(n) - 1) \end{cases}$$

$$0 \leq \chi_n'(t) \leq 1, \;\; \chi_n(0) = 0.$$

So

$$\chi_n(t) = \begin{cases} t, & t \leq max(n+1, M_n) \\ B_n, & t \geq M_n(l(n) - 1) \end{cases}$$

$B_n > max(n+1, M_n)$. We may assume that

$$|\chi_n^{(s)}(t)| = |(\chi_n')^{(s-1)}(t)| \leq \frac{A}{[(l(n)-1)M_n - max(n+1, M_n)]^{s-1}}, \;\; s = 2, \dots, k$$

(Note that we may choose

$$\chi_n(t) = \tilde{\chi}\left(\frac{t - max(n+1, M_n)}{(l(n)-1)M_n - max(n+1, M_n)}\right)$$

where $\tilde{\chi}$ is a fixed function such that

$$\tilde{\chi}'(s) = \begin{cases} 1, & s \le 0 \\ 0, & s \ge 1, \end{cases}$$

$$0 \le \tilde{\chi}'(s) \le 1, \ \ \chi(0) = 0).$$

So $\Psi_{n+1}|_{\Omega_n} = \Psi_n$, $\Psi_{n+1} = B_n > max(n+1, M_n)$ beyond the points $b_i$,

$$\sup_{\substack{x\in\breve{\omega}_{n+1}\\ |\alpha|=k}} |\Psi^{(\alpha)}_{n+1}(x)| = \sup_{\substack{x\in\breve{\omega}_{n+1}\\ |\alpha|=k}} |(\chi_n(\mu_n(x)))^{(\alpha)}(x)|.$$

Choosing $l(n)$ to be sufficiently large, we may make all derivatives $\chi_n^{(s)}$ $(s = 2, 3, \dots, k)$ very small, so

$$\sup_{\substack{x\in\breve{\omega}_{n+1}\\ |\alpha|=l}} |(\chi_n(\mu_n(x)))^{(\alpha)}(x)| \le$$

$$\le \sup_{\substack{x\in\breve{\omega}_{n+1}\\ |\alpha|=l}} |\chi_n'(\mu_n(x))\mu_n^{(\alpha)}(x)| + \frac{1}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right) h_l \le$$

$$\le h_l\left[1 - \frac{1}{n} + \frac{3}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right] + \frac{1}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right) h_l = h_l\left[1 - \frac{1}{n} + \frac{7}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right].$$

So

$$\sup_{\substack{x\in\breve{\omega}_{n+1}\\ |\alpha|=l}} |\Psi^{(\alpha)}_{n+1}(x)| \le h_l\left[1 - \frac{1}{n} + \frac{7}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right].$$

The function may be obviously extended to the remaining part of $\omega_{n+1}$ (beyond the points $b_i$) just by putting $\Psi_{n+1}$ there to be constant $B_n \ge max(n+1, M_n) \ge n+1$. This function is obviously extendible to a narrow neighborhood of $\breve{\omega}_{n+1} \cup \omega_{n+1}$. We may assume this neighborhood to have a real analytic boundary and we put $\bar{\Omega}_{n+1}$ to be the closure of this neighborhood. Choosing the neighborhood $\Omega_{n+1}$ to be sufficiently narrow we may ensure that the extended function (we still denote it $\Psi_{n+1}$) satisfies the estimate

$$\sup_{\substack{x\in\Omega_{n+1}\\ |\alpha|=l}} |\Psi^{(\alpha)}_{n+1}(x)| \le h_l\left[1 - \frac{1}{n} + \frac{7}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right] + \frac{1}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right) h_l = h_l\left[1 - \frac{1}{n+1}\right].$$

So we have constructed: a domain $\Omega_{n+1}$ with a real analytic boundary, $\Omega_{n+1} \supset \bar{\Omega}_n$, $\Omega_{n+1}$ is contained in the circle $C_0$, a finite $\frac{1}{n}$-net $T_n \subset \bar{\Omega}_n$, $T_n \supset T_{n-1}$, a smooth function $\Psi_{n+1}$ on $\bar{\Omega}_{n+1}$, such that

$$\sup_{\substack{x\in\bar\Omega_{n+1}\\ |\alpha|=l}} |\Psi^{(\alpha)}_{n+1}(x)| \le h_l\left[1-\frac{1}{n+1}\right],\ 1\le l\le k,$$

$$\Psi_{n+1}(b_i) = B_n \ge n+1.$$

So the properties $(\Omega_{n+1}, 1-3)$ are checked. Let us check the property $(\Omega_{n+1}$,4), showing that

$$\|m^k(z-x;x)-m^k(z-y;y)\|_{M^\infty_{k+1}(\Omega_{n+1})} \le C(k)\int_{[xy]} |z-w|^k d|w|$$

for any $z\in Z_n$ and for any $x,y\in T_n$. Really, $[xy]\subset\omega_{n+1}$. If $[xy]\subset\bar\Omega_n$, then $[xy]\subset\Omega_{n+1}$ and the estimate is obvious. So, let $[xy]\not\subset\bar\Omega_n$. Then $[xy]\backslash\bar\Omega_n$ consists of several segments. Each of these segments is interrupted twice and the above described construction, consisting of $|Z_n|$ labyrinths, is inserted into each interruption. The set $Z_n$ consists of the elements $z_1,\dots,z_p$, $p=|Z|$. Let $z=z_i$. Let

$$\mathcal{L}_{a_i^1,b_i^1,A_n}(z_i),\ \mathcal{L}_{a_i^2,b_i^2,A_n}(z_i),\dots,\mathcal{L}_{a_i^q,b_i^q,A_n}(z_i)$$

be the labyrinths of these constructions, listed in the order they are met on $[xy]$, beginning from $x$. Consider the decomposition

$$m^k(z_i-x;x)-m^k(z_i-y;y)=$$

$$[m^k(z_i-x;x)-m^k(z_i-a_i^1;a_i^1)]+[m^k(z_i-a_i^1;a_i^1)-m^k(z_i-b_i^1;b_i^1)]+$$

$$+[m^k(z_i-b_i^1;b_i^1)-m^k(z_i-b_i^2;b_i^2)]+[m^k(z_i-b_i^2;b_i^2)-m^k(z_i-a_i^2;a_i^2)]+$$

$$+[m^k(z_i-a_i^2;a_i^2)-m^k(z_i-a_i^3;a_i^3)]+\dots+[m^k(z_i-a_i^q;a_i^q)-m^k(z_i-y;y)].$$

So, using the estimate for labyrinths, we obtain

$$\|m^k(z_i-x;x)-m^k(z_i-y;y)\|_{M_\infty^{k+1}(\Omega_{n+1})}\le$$

$$\le\|m^k(z_i-x;x)-m^k(z_i-a_i^1;a_i^1)\|_{M_\infty^{k+1}(\Omega_{n+1})}+$$

$$+\sum_{r=1,3,5,\dots}\{\|m^k(z_i-a_i^r;a_i^r)-m^k(z_i-b_i^r;b_i^r)\|_{M_\infty^{k+1}(\Omega_{n+1})}+$$

$$+\|m^k(z_i-b_i^r;b_i^r)-m^k(z_i-b_i^{r+1};b_i^{r+1})\|_{M_\infty^{k+1}(\Omega_{n+1})}+$$

$$+\|m^k(z_i-b_i^{r+1};b_i^{r+1})-m^k(z_i-a_i^{r+1};a_i^{r+1})\|_{M_\infty^{k+1}(\Omega_{n+1})}+$$

$$+\|m^k(z_i-a_i^{r+1};a_i^{r+1})-m^k(z_i-a_i^{r+2};a_i^{r+2})\|_{M_\infty^{k+1}(\Omega_{n+1})}\}+$$

$$+\|m^k(z_i-a_i^q;a_i^q)-m^k(z_i-y;y)\|_{M_\infty^{k+1}(\Omega_{n+1})}\le$$

$$\le\int_{\widehat{xa_i^1}}|z_i-w|^kd|w|+\sum_{r=1}^{q-1}\{C(k)\int_{[a_i^rb_i^r]}|z_i-w|^kd|w|+\int_{\widehat{b_i^rb_i^{r+1}}}|z_i-w|^kd|w|+$$

$$+C(k)\int_{[a_i^{r+1}b_i^{r+1}]}|z_i-w|^kd|w|+\int_{\widehat{a_i^{r+1}a_i^{r+2}}}|z_i-w|^kd|w|+\int_{\widehat{a_i^qy}}|z_i-w|^kd|w|\}.$$

Here we remark that the arcs $\widehat{xa_i}$, $\widehat{b_i^r b_i^{r+1}}$, $\widehat{a_i^{r+1} a_i^{r+2}}$, $\widehat{a_i^q y}$ are "almost straight", i.e. their lengths are not greater than twice the related distances, so we may estimate the above sum by

$$C(k)\int_{[xy]} |z_i - w|^k d|w|$$

and the property ($\Omega_{n+1}$,4) is verified. So the required domain is constructed.

## §3. Labyrinths

**3.1 Elementary and standard labyrinths.**

Elementary labyrinths are special systems of horizontal and vertical segments, equipped with special systems of points.

Consider a Cartesian coordinate system. We identify the point $(x, y)$ with the vector $xe_1 + ye_2$, where $e_1, e_2$ are the unit vectors of the coordinate axis. First we describe the mentioned systems of points. Fix a natural $k$ and consider the points

$$v_{0,j} = (0, \frac{j}{k+1}),\ j = 1, 2, \ldots, k+1.$$

Fix $\alpha$, $0 < \alpha < 1$ (further we shall choose $\alpha$ very close to 1), and fix an odd $L \in \mathbb{N}$ such that $\alpha^k(1 - \alpha^{(L-1)k}) > \frac{1}{2}$. Consider the points

$$v_{s,j} = (0, (-\alpha)^s \frac{j}{k+1}),\ \ j = 1, 2, \ldots, k+1;\ \ s = 0, 1, \ldots, L;$$

these points are called "the $s$-th generation vertices". Additionally we assume that all numbers $\nu_{s,j} = (-\alpha)^s \frac{j}{k+1}$ ($j = 1, 2, \ldots, k+1,\ s = 0, 1, \ldots, L$) are pairwise distinct (for example, we may choose $\alpha$ to be transcendental). Next, fix a small $\varepsilon > 0$, such that $4(k+1)L\varepsilon < 1$. Consider the horizontal intervals

$$H(s, j) = \{(x, (-\alpha)^s \frac{j}{k+1}) : -\varepsilon\alpha^s \le x \le 0\}$$

– we call them "the $s$-th generation horizontals". Obviously, $v_{s,j} \in H(s, j)$, $v_{s,j}$ are the right ends of the corresponding horizontals $H(s, j)$. Let $l_{s,j}$ denote the left end of the horizontal $H(s, j)$, $l_{s,j} = (-\varepsilon\alpha^s, (-\alpha)^s \frac{j}{k+1})$. The length of horizontals of the next generation is smaller then the length of horizontals of a given generation.

Consider the vertical interval

$$\tilde{V}(L) = \{(-\varepsilon\alpha^L, y) : (-\alpha)^L = -\alpha^L \le y \le \alpha^{L-1} = (-\alpha)^{L-1}\}.$$

This vertical contains all $l_{L,j}$ $(1 \le j \le k+1)$ and it intersects many horizontals of previous generations. We want to improve $\tilde{V}(L)$ in such a way that it will **not** intersect horizontals of the generations, previous to the $L$-th one (except of the horizontal $H(L-1, k+1)$), and its length will not change too much.

Let $\delta$ denote a positive number such that the intervals $[\nu_{s,j} - \delta, \nu_{s,j} + \delta]$ are pairwise disjoint ($s = 0, 1, \ldots, L,\quad j = 1, 2, \ldots, k+1$). Now let us erase the following subintervals of $\tilde{V}(L)$ :

$$\omega(L; s, j) = \{(-\varepsilon\alpha^L, y) : \nu_{s,j} - \delta \le y \le \nu_{s,j} + \delta\}$$

$$(s,j) : H(s,j) \bigcap \tilde{V}(L) \neq \emptyset,\ s \leq L-1,\ (s,j) \neq (L-1, k+1)$$

and, instead of them, let us paste in the following constructions:

$$ps(L; s, j) = \{(x, \nu_{s,j} - \delta),\ -\varepsilon\alpha^L \leq x \leq \frac{\varepsilon\alpha^L}{2}\}$$

$$\bigcup \{(\frac{\varepsilon\alpha^L}{2}, y),\ \nu_{s,j} - \delta \leq y \leq \nu_{s,j} + \delta\}$$

$$\bigcup \{(x, \nu_{s,j} + \delta), -\varepsilon\alpha^L \leq x \leq \frac{\varepsilon\alpha^L}{2}\}.$$

Fig. 2

So, let

$$V(L) = \left( \tilde{V}(L) \setminus \bigcup_{\substack{(s,j):H(s,j)\cap\tilde{V}(L)\neq\emptyset \\ s\leq L-1 \\ (s,j)\neq(L-1,k+1)}} er(L; s, j) \right) \bigcup \left( \bigcup_{\substack{(s,j):H(s,j)\cap\tilde{V}(L)\neq\emptyset \\ s\leq L-1 \\ (s,j)\neq(L-1,k+1)}} ps(L; s, j) \right)$$

Obviously, $V(L)$ does not intersect horizontals of the generations previous to the $L$-th one, except of the horizontal $H(L-1, k+1)$. Its length does not exceed

$$\alpha^L + \alpha^{L-1} + 2 \cdot (k+1) \cdot L \cdot 2\varepsilon \cdot \alpha^L \leq \alpha^{L-1} + \alpha^L(1 + 4(k+1)L\varepsilon) \leq$$

$$\leq \alpha^{L-1} + \alpha^L 2 \leq 2(\alpha^{L-1} + \alpha^L) = 2 \text{ length } \tilde{V}(L) < 4\alpha^L.$$

Now consider the vertical

$$\tilde{V}(L-1) = \{(-\varepsilon\alpha^{L-1}, y) : y \text{ between } (-\alpha)^{L-2} \text{ and } (-\alpha)^{L-1}\}$$

This vertical contains all $l_{L-1,j}$ $(1 \leq j \leq k+1)$ and it intersects many horizontals of the **previous** generations (and does not intersect horizontals of the next generation). We again wish to improve $\tilde{V}(L-1)$ in such a way that it will **not** intersect horizontals of generations other than the $(L-1)$-th one, except of the horizontal $H(L-2, k+1)$, it will not intersect $V(L)$, and its length will not change too much.

Consider the following subintervals of $\tilde{V}(L-1)$

$$er(L-1;s,j) = \{(-\varepsilon\alpha^{L-1}, y) : \nu_{s,j} - \frac{\delta}{2} \leq y \leq \nu_{s,j} + \frac{\delta}{2}\}.$$

$$(s,j) : H(s,j) \cap \tilde{V}(L-1) \neq \emptyset, \;\; s \leq L-2, \;\; (s,j) \neq (L-2, k+1).$$

Consider the following constructions:

$$\begin{aligned} ps(L-1;s,j) = &\{(x, \nu_{s,j} - \frac{\delta}{2}) : -\varepsilon\alpha^{L-1} \leq x \leq \frac{\varepsilon\alpha^L}{4}\} \bigcup \\ &\{(\frac{\varepsilon\alpha^L}{4}, y) : \nu_{s,j} - \frac{\delta}{2} \leq y \leq \nu_{s,j} + \frac{\delta}{2}\} \bigcup \\ &\{(x, \nu_{s,j} + \frac{\delta}{2}) : -\varepsilon\alpha^{L-1} \leq x \leq \frac{\varepsilon\alpha^L}{4}\}. \end{aligned}$$

Let

$$V(L-1) =$$

$$\left( \tilde{V}(L-1) \setminus \bigcup_{\substack{(s,j):H(s,j)\cap\tilde{V}(L-1)\neq\emptyset \\ s\leq L-2 \\ (s,j)\neq(L-2,k+1)}} er(L-1;s,j) \right) \bigcup \left( \bigcup_{\substack{(s,j):H(s,j)\cap\tilde{V}(L-1)\neq\emptyset \\ s\leq L-2 \\ (s,j)\neq(L-2,k+1)}} ps(L-1;s,j) \right)$$

Obviously, $V(L-1)$ does not intersect $V(L)$ and does not intersect the horizontals of the generations $1, 2, \ldots, L-2$, except of the horizontal $H(L-2, k+1)$. Its length does not exceed

$$\alpha^{L-2} + \alpha^{L-1} + 2(k+1)(L-1)2\varepsilon\alpha^{L-1} \leq$$

$$\leq \alpha^{L-2} + \alpha^{L-1}(1 + 4(k+1)L\varepsilon) \leq 2 \text{ length } \tilde{V}(L-1) < 4\alpha^{L-1}.$$

Going on, we construct the sets

$$V(L-1),\, V(L-3), \ldots, V(1) \text{ and } V(0) = \{(-\varepsilon, y) : \; 0 \leq y \leq 1\}$$

with the following properties:

(i) $V(s)$ connects all $H(s,j)$ to $H(s-1,k+1)$ ($j = 1, \ldots, k+1$); $s = 1, 2, \ldots, L$; $V(0)$ connects all $H(0,j)$ to $H(0,k+1)$;

(ii) the length of $V(s)$ does not exceed $4\alpha^s$;

(iii) Consider $v_{s-1,j}$ and $v_{s,p}$ ($1 \leq j, p \leq k+1$, $1 \leq s \leq L$). There is a path connecting $v_{s-1,j}$ and $v_{s,p}$ (the path consists of $H(s-1,j)$, a part of $V(s-1)$, a part of $H(s-1,k+1)$, a part of $V(s)$ and $H(s,p)$). Its length does not exceed

$$(\varepsilon\alpha^{s-1}) + 4\alpha^{s-1} + (\varepsilon\alpha^{s-1}) + 4\alpha^s + (\varepsilon\alpha^s) \leq$$

$$\leq 5(\alpha^{s-1} + \alpha^s) \leq 5(k+1)\left(\frac{\alpha^{s-1} + \alpha^s}{k+1}\right) \leq$$

$$\leq 5(k+1)|$$

Now we introduce an **elementary labyrinth**:

$$L(\alpha,\varepsilon)=$$

$$=\bigcup_{s=0}^{L}V(s)\bigcup_{\substack{0\le s\le L\\ 1\le j\le k+1}}H(s,j)\bigcup\{(x,\frac{(-\alpha)^L}{k+1}):0\le x\le\varepsilon\alpha^L\}\bigcup\{(\varepsilon\alpha^L,y):0\ge y\ge-1\}$$

Note that all $v_{s,j}\in L(\alpha,\varepsilon)$.

Choose $\gamma>0,\rho>0$. Consider the sets

$$L_i(\alpha,\varepsilon,\gamma,\rho)=\{(x+\frac{i}{k+1}\gamma,\rho y):(x,y)\in L(\alpha,\varepsilon\rho)\},\ \ i=1,2,\dots,k+1,$$

$\varepsilon$ is chosen so small that $L_i(\alpha,\varepsilon,\gamma,\rho)$ are pairwise disjoint and $4L(k+1)\varepsilon\rho<1$. Now we introduce a **standard labyrinth** $L(\alpha,\varepsilon;\gamma,\rho)$ :

$$L(\alpha,\varepsilon;\gamma,\rho)=\bigcup_{i=1}^{k+1}L_i(\alpha,\varepsilon;\gamma,\rho)\bigcup\{(x,\rho):0\le x\le\gamma\}$$

$$\bigcup\{(0,y):0\le y\le\rho\}\bigcup\{(x,-\rho):0\le x\le\gamma\,\frac{k+2}{k+1}\}\bigcup\{(\gamma\,\frac{k+2}{k+1},y):0\ge y\ge-\rho\}.$$

Further we omit $\varepsilon$ in the notation, assuming only that $\varepsilon$ is small enough. Note that any path in $L(\alpha,\gamma,\rho)$, joining $(0,0)$ and $(\gamma\frac{k+2}{k+1},0)$, must pass through one of the elementary labyrinths $L_i(\alpha,\gamma,\rho)$. The point $(0,0)$ is called the **source** of the labyrinth $L(\alpha,\gamma,\rho)$, the point $(\gamma\,\frac{k+2}{k+1},0)$ is called the **sink** of the labyrinth.

### 3.2. Decomposition.

Consider the measure

$$m^k(\Lambda e_1;0)-m^k(\Lambda e_1-b;b)$$

where $e_1$ is the vector $\{1,0\}$, $0=(0,0)$ is the source and $b$ is the sink point (and the corresponding vector) $(\gamma\,\frac{k+2}{k+1},0)$.

We present a special decomposition of this measure:

Let $v^i_{sj}=\rho v_{sj}+\frac{i}{k+1}\gamma e_1,\ s=0,1,\dots,L,\ i,j=1,2,\dots,k+1$

$$m^k(\Lambda e_1;0)-m^k(\Lambda e_1-b;b)=$$

$$=\sum_{1\le i,j\le k+1}x_iy_j[m^k(v^i_{0j};0)-m^k(0;v^i_{0j})]+$$

$$+\sum_{1\le i,j,p\le k+1}x_iz_{jp}[m^k(0;v^i_{0j})-m^k(v^i_{0j}-v^i_{1p};v^i_{1p})]+$$

$$+\sum_{1\le i,j,p\le k+1}x_iz_{pj}[m^k(0;v^i_{1p})-m^k(v^i_{1p}-v^i_{2j};v^i_{2j})]+\dots$$

$$\dots+\quad\sum\quad x_iz_{jp}[m^k(0;v^i_{sj})-m^k(v^i_{sj}-v^i_{(s+1),p};v^i_{(s+1),p})]+\dots$$

$$+\cdots \sum_{1\le i,j\le k+1} x_i y_j [m^k(0; v^i_{Lj}) - m^k(v^i_{Lj}-b; b)].$$

The numbers $x_i, y_j, z_{jp}$ will be chosen below.

This decomposition is valid provided the following equalities hold:

(A) $$\Lambda^q e_1^{\otimes q} = \sum_{1\le i,j\le k+1} x_i y_j (v^i_{0j})^{\otimes q},\ q=0,1,\dots,k$$

(B) $$\sum_{1\le p\le k+1} z_{jp} = y_j,\ j=1,\dots,k+1$$

(C) $$\sum_{1\le j\le k+1} z_{pj} = \sum_{1\le j\le k+1} z_{jp},\ \ p=1,2,\dots,k+1$$

(D.01) $$0=\sum_{1\le j\le k+1} z_{jp}(v^i_{0j}-v^i_{1p})^{\otimes q},\ \ q=1,\dots,k;\ \ i,p=1,2,\dots k+1$$

(D.12) $$0=\sum_{1\le p\le k+1} z_{pj}(v^i_{1p}-v^i_{2j})^{\otimes q},\ \ q=1,\dots,k;\ \ i,j=1,2,\dots,k+1$$

$$\cdots\cdots\cdots\cdots\cdots$$

(D.(L-1)L) $$0=\sum_{1\le p\le k+1} z_{pj}(v^i_{L-1,p}-v^i_{Lj})^{\otimes q},\ \ q=1,\dots,k;\ \ i,j=,2,\dots,k+1$$

(E) $$(\Lambda e_1 - b)^{\otimes q} = \sum_{1\le i,j\le k+1} x_i y_j (v^i_{Lj}-b)^{\otimes q},\ \ q=0,\dots,k$$

So, we write a system of equations for $x_i, y_j, z_{jp}$. Since we are working in the **symmetric** tensor power of $\mathbb{R}^2$, (A) gives:

$$\Lambda^q e_1^{\otimes q} = \sum_{1\le i,j\le k+1} x_i y_j \left(\rho\nu_{0j} e_2 + \frac{i}{k+1}\gamma e_1\right)^{\otimes q} =$$

$$= \sum_{\substack{1\le i,j\le k+1\\ 0\le r\le q}} x_i y_j \binom{q}{r} (e_2^{\otimes(q-r)}\otimes e_1^{\otimes r}) \left(\frac{i\gamma}{k+1}\right)^r \nu_{0j}^{q-r}\rho^{q-r}$$

or

$$\Lambda^q = \sum_{1\le i,j\le k+1} x_i y_j \left(\frac{i\gamma}{k+1}\right)^q,\ \ q=0,1,\dots,k$$

$$0=\quad \sum \quad x_i y_j \binom{q}{r}\left(\frac{i\gamma}{k+1}\right)^r \nu_{0,j}^{q-r}\rho^{q-r},\ \ q=0,1,\dots,k;\ r=0,1,\dots,q-1$$

Let us put $\sum_{1\le j\le k+1} y_j = 1$. Then $x_i$ are uniquely determined from the equations

$$(A_x) \qquad (\frac{k+1}{\gamma}\Lambda)^q = \sum_{1\le i\le k+1} x_i i^q, \;\; q = 0,1,\dots,k$$

(this is a Vandermond system with the matrix $V = (i^q)$, $1 \le i \le k+1$, $0 \le q \le k$). Then $y_j$ are uniquely determined by the equations

$$\sum_j y_j \nu_{0j}^{q-r} = 0, \;\; q-r = 1,2,\dots,k$$
$$\sum_j y_j = 1$$

or

$$(A_y) \qquad \begin{aligned} \sum_j y_j j^q = 0, \;\; q = 1,2,\dots,k \\ \sum_j y_j = 1 \end{aligned}$$

(this is again a Vandermond system with the same matrix $V$).

In order to deal with the equalities (D.01), (D.12), ..., (D.(L-1)L), we note that

$$v_{sp}^i - v_{(s+1),j}^i = \rho\nu_{sp}e_2 + \frac{i}{k+1}\gamma e_1 - \rho\nu_{s+1,j}e_2 - \frac{i}{k+1}\gamma e_1 =$$

$$\rho[(-\alpha)^s\frac{p}{k+1} - (-\alpha)^{s+1}\frac{j}{k+1}]e_2 = \frac{\rho(-\alpha)^s}{k+1}(p+j\alpha)e_2.$$

So the equalities (D.01), (D.12), ..., (D.(L-1)L) simply mean that

$$\sum_{1\le p\le k+1} z_{pj}\left[\frac{\rho(-\alpha)^s}{k+1}(p+j\alpha)\right]^q = 0 \;\; (q = 1,2,\dots,k; \; j = 1,2,\dots,k+1)$$

or

$$\sum_{1\le p\le k+1} z_{pj}(p+j\alpha)^q = 0, \;\; (q = 1,2,\dots k; \; j = 1,2,\dots,k+1)$$

Recall that

$$\sum_{1\le p\le k+1} z_{pj} = \sum_{1\le p\le k+1} z_{jp} = y_j, \;\; j = 1,2,\dots k+1.$$

So,

$$0 = \sum_p z_{pj}(p+j\alpha) = \sum_p z_{pj}p + \alpha j\sum_p z_{pj} = \sum_p pz_{pj} + \alpha jy_j,$$

$$0 = \sum_p z_{pj}(p+j\alpha)^2 = \sum_p z_{pj}p^2 + 2j\alpha\sum_p z_{pj}p + \alpha^2 j^2\sum_p z_{pj} =$$

$$= \sum z_{pj}p^2 + 2j\alpha(-\alpha jy_j) + \alpha^2j^2y_j = \sum z_{pj}p^2 - j^2\alpha^2y_j,$$

$$0 = \sum_p z_{pj}(p + j\alpha)^3 =$$

$$= \sum_p z_{pj} p^3 + 3j\alpha \sum_p z_{pj} p^2 + 3j^2\alpha^2 \sum_p z_{pj} p + j^3\alpha^3 \sum_p z_{pj} =$$

$$= \sum_p z_{pj} p^3 + 3\alpha^3 j^3 y_j - 3\alpha^3 j^3 y_j + j^3\alpha^3 y_j = \sum_p z_{pj} p^3 + j^3\alpha^3 y_j$$

$$\dots\dots\dots\dots\dots$$

So we get:

$$\sum_p z_{jp} = y_j \tag{B}$$

and

$$\begin{aligned} \sum_p z_{pj} &= y_j \\ \sum_p p z_{pj} &= (-\alpha j) y_j \\ \sum_p p^2 z_{pj} &= (-\alpha j)^2 y_j \\ &\dots\dots\dots\dots \\ \sum_p p^k z_{pj} &= (-\alpha j)^k y_j \end{aligned} \tag{D}$$

Introduce the matrices

$$Z = (z_{pj})_{p,j=1,\dots,k+1},$$

$$Diag(y_1, \dots, y_{k+1}) = (\delta_{ij} y_j)_{i,j=1,\dots,k+1}$$

$$Diag(p) = (\delta_{ij} p^i)_{i,j=1,\dots,k+1},$$

$$\mathbb{I} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \ \mathbb{I}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_{k+1} \end{pmatrix}, \ Y = \begin{pmatrix} y_1 \\ \vdots \\ y_{k+1} \end{pmatrix},$$

Then we have

$$VX = Diag(\frac{(k+1)\Lambda}{\gamma})\,\mathbb{I} \tag{$A_x$}$$

$$VY = \mathbb{I} \tag{$A_y$}$$

(B) $$Z\mathbb{I} = Y$$

(D) $$VZ = Diag(-\alpha)VDiag(y_1, \ldots, y_{k+1}).$$

So we obtain

$$X = V^{-1}Diag(\frac{(k+1)\Lambda}{\gamma})\,\mathbb{I}$$

$$Y = V^{-1}\mathbb{I}_1$$

$$Z = V^{-1}Diag(-\alpha)VDiag(y_1, \ldots, y_{k+1}).$$

Let us check that $Z\mathbb{I} = Y$ :

$$Z\mathbb{I} = V^{-1}Diag(-\alpha)VDiag(y_1, \ldots, y_{k+1})\mathbb{I} =$$

$$= V^{-1}Diag(-\alpha)VY = V^{-1}Diag(-\alpha)\mathbb{I}_1 = V^{-1}\mathbb{I}_1 = Y.$$

The only thing remained to be checked is (E). As $b = (\gamma\,\frac{k+2}{k+1})e_1 = \tilde{\gamma}e_1$, so we must check that

$$(\Lambda - \tilde{\gamma})^q e_1^{\otimes q} = \sum_{1\le i,j\le k+1} x_i y_j \left(\frac{\gamma i}{k+1}e_1 + (-\alpha)^L \frac{j}{k+1}e_2 - \tilde{\gamma}e_1\right)^{\otimes q}.$$

Calculate the right hand part:

$$\sum_{\substack{1\le i,j\le k+1\\0\le r\le q}} x_i y_j \binom{q}{r}\left(\frac{\gamma i}{k+1} - \tilde{\gamma}\right)^r (-\alpha)^{L(q-r)}\frac{j^{q-r}}{(k+1)^{q-r}}e_1^{\otimes r}\otimes e_2^{\otimes(q-r)} =$$

$$= \sum_{1\le i,j\le k+1} x_i y_j \left(\frac{\gamma i}{k+1} - \tilde{\gamma}\right)^q e_1^{\otimes q} +$$

$$+ \sum_{\substack{1\le i,j\le k+1\\0\le r\le q-1}} x_i y_j \binom{q}{r}\left(\frac{\gamma i}{k+1} - \tilde{\gamma}\right)^r (-\alpha)^{L(q-r)}\frac{j^{q-r}}{(k+1)^{q-r}}\,e_1^{\otimes r}\otimes e_2^{\otimes(q-r)} =$$

$$= \left[(\sum_{1\le j\le k+1} y_j)\sum_{0\le r\le q}\binom{q}{r}(-\tilde{\gamma})^{q-r}(\sum_{1\le i\le k+1} x_i i^r)\,\frac{\gamma^r}{(k+1)^r}\right]e_1^{\otimes q} +$$

$$+ \sum_{0\le r\le q-1}(\sum_{1\le j\le k+1} y_j j^{q-r})\times$$

$$\left[\sum_{1\le i\le k+1} x_i \binom{q}{r}\left(\frac{\gamma i}{k+1} - \tilde{\gamma}\right)^r (-\alpha)^{L(q-r)}\frac{1}{(k+1)^{q-r}}\,e_1^{\otimes r}\otimes e_2^{\otimes(q-r)}\right].$$

The first expression here equals

$$1\cdot\left[\sum_{0\le r\le q}\binom{q}{r}(-\tilde\gamma)^{q-r}\left(\frac{\Lambda(k+1)}{\gamma}\right)^r\frac{\gamma^r}{(k+1)^r}\right]e_1^{\otimes q}=(\Lambda-\tilde\gamma)^q e_1^{\otimes q}.$$

(see $(A_x)$), the second is zero (see $(A_y)$), and (E) is checked.

So, the decomposition is valid, provided

$$X=V^{-1}\begin{pmatrix}1\\ \frac{(k+1)\Lambda}{\gamma}\\ \vdots\\ (\frac{(k+1)\Lambda}{\gamma})^k\end{pmatrix},\quad Y=V^{-1}\mathbb{I}_1,$$

$$Z=V^{-1}Diag(-\alpha)VDiag(y_1,\dots,y_{k+1}).$$

Note that $y_i$ are constants, $z_{jp}$ are polynomials in $\alpha$ and do not depend upon $\gamma$, $\rho$; further, $x_i$ do not depend upon $\alpha$, $\rho$ and yield the following estimate with $C$ depending only on $k$ :

$$|x_i|\le C\left(\frac{\Lambda}{\gamma}\right)^k$$

for large $\frac{\Lambda}{\gamma}$.

**Remark.** E.M. Dyn'kin observed that the above decomposition may be described as a version of the Lagrange interpolation process: a polynomial is first interpolated with respect to $x$ at the nodes $(\frac{i}{k+1},0)$ and then each of the obtained polynomials is interpolated with respect to $y$ at the nodes $(\frac{i}{k+1},\nu_{s,j})$.

**3.3. Estimate.**

Consider any domain $\Sigma$ containing the labyrinth $L(\alpha,\gamma,\rho)$. Let us estimate the norm

$$\|m^k(\Lambda e_1;0)-m^k(\Lambda e_1-b;b)\|_{M^\infty_{k+1}(\Sigma)},$$

assuming that $\frac{\Lambda}{\gamma}$ is large.

First we need to estimate the following expressions:

$$\|m^k(v^i_{0j};0)-m^k(0;v^i_{0j})\|_{M^\infty_{k+1}(\Sigma)},$$

$$\|m^k(0;v^i_{sj})-m^k(v^i_{sj}-v^i_{(s+1);p},v^i_{(s+1),p})\|_{M^\infty_{k+1}(\Sigma)},$$

$$\|m^k(0;v^i_{Lj})-m^k(v^i_{Lj}-b;b)\|_{M^\infty_{k+1}(\Sigma)}.$$

Our estimates of these expressions are based on the following facts:

(a) there exists a path in $L(\alpha,\gamma,\rho)$, connecting 0 to $v^i_{0j}$ of length not exceeding $2|v^i_{0j}|$;

(b) there exists a path in $L(\alpha,\gamma,\rho)$, connecting $v^i_{sj}$ to $v^i_{(s+1),p}$ of length not exceeding $5(k+1)|v^i_{sj}-v^i_{(s+1),p}|$;

(c) there exists a path in $L(\alpha,\gamma,\rho)$, connecting $v^i_{Lj}$ to $b$ of length not exceeding

First, let us check (b). It is equivalent to the mentioned fact that there exists a path in $L(\alpha)$, connecting $v_{sj}$ to $v_{(s+1),p}$ of length not exceeding

$$5(k+1)\,|v_{sj}-v_{(s+1),p}|.$$

(Note that $L_i(\alpha,\gamma,\rho)$ is obtained from $L(\alpha)$ by scaling by the factor $\rho$ and by a shift).

Next, the path, required in (a), may be obtained as follows:

$$\{(0,y):0\le y\le\rho\}\bigcup\{(x,\rho):0\le x\le\frac{\gamma i}{k+1}-\varepsilon\rho\}$$

$$\bigcup\{(\frac{\gamma i}{k+1}-\varepsilon\rho,y):\rho\frac{j}{k+1}\le y\le\rho\}\bigcup\{(x,\rho\frac{j}{k+1}):\frac{\gamma i}{k+1}-\varepsilon\rho\le x\le\frac{\gamma i}{k+1}\}.$$

Its length does not exceed

$$\rho+\frac{\gamma i}{k+1}+\rho=\frac{\gamma i}{k+1}\left(1+\frac{\rho}{\gamma}\cdot\frac{2(k+1)}{i}\right).$$

Taking $\frac{\rho}{\gamma}$ such that $\frac{\rho}{\gamma}\cdot 2(k+1)<1$, we obtain that the length of the path does not exceed

$$2\frac{\gamma i}{k+1}\le 2|v^i_{0,j}|.$$

And, finally, the path required in (c), may be obtained as follows:

$$\{\text{ path from } v^i_{Lj} \text{ to } v^i_{L,1}\}\bigcup\{(x+\frac{\gamma i}{k+1},\frac{-\rho\alpha^L}{k+1}):0\le x\le\varepsilon\rho\alpha^L\}$$

$$\bigcup\{(\varepsilon\rho\alpha^L+\frac{\gamma i}{k+1},y):\frac{-\rho\alpha^L}{k+1}\ge y\ge-\rho\}$$

$$\bigcup\{(x,-\rho):\varepsilon\rho\alpha^L+\frac{\gamma i}{k+1}\le x\le\gamma\frac{k+2}{k+1}\}$$

$$\bigcup\{(\gamma\frac{k+2}{k+1},y):-\rho\le y\le 0\}.$$

Its length does not exceed

$$\rho(\text{ length of } V_L)+\varepsilon\rho\alpha^L+\rho+\gamma\,\frac{k+2-i}{k+1}+\rho\le$$

$$\le\rho\cdot 4\alpha^L+\varepsilon\rho\alpha^L+2\rho+\gamma\,\frac{k+2-i}{k+1}\le$$

$$\le 2\rho+\rho+\rho+\gamma\,\frac{k+2-i}{k+1}=\gamma\,\frac{k+2-i}{k+1}\left(1+\frac{\rho}{\gamma}\frac{5(k+1)}{(k+2-i)}\right).$$

We may assume that

$$\frac{\rho 5(k+1)}{\gamma(k+2-i)}\le 1,\quad i=1,2,\dots,k+1,$$

so the length of the path does not exceed

$$2\gamma \frac{k+2-i}{k+1} \le 2|v^i_{Lj} - b|.$$

Let us note that if there exists a polygonal path in $\Sigma$, connecting $x$ and $y$, such that its length is not exceeding $C|x-y|$, then

$$\|m^k(x-y;y) - m^k(0;x)\|_{M^\infty_{k+1}(\Sigma)} \le C^{k+1}|x-y|^{k+1}$$

Really there exist $x_0 = y, x_1, \dots, x_n = x$, $x_i \in \Sigma$, $[x_i x_{i+1}] \subset \Sigma$ $(i = 0, 1, \dots, n-1)$ $\sum_{i=0}^{n-1} |x_{i+1} - x_i| \le C|x-y|$. Then

$$m^k(x-y;y) - m^k(0;x) = \sum_{i=0}^{n-1} (m^k(x-x_i;x_i) - m^k(x-x_{i+1}, x_{i+1}))$$

so

$$\|m^k(x-y;y) - m^k(0;x)\|_{M^\infty_{k+1}(\Sigma)} \le$$

$$\le \sum_{i=0}^{n-1} \|m^k(x-x_i;x_i) - m^k(x-x_{i+1};x_{i+1})\|_{M^\infty_{k+1}(\Sigma)} \le$$

$$\le \sum_{i=0}^{n-1} \int_{[x_i x_{i+1}]} |x-w|^k d|w| \le C \max_{w \in \cup_{i=0}^{n-1}[x_i, x_{i+1}]} |x-w|^k \cdot |x-y| =$$

$$= C \max_{0 \le i \le n} |x-x_i|^k |x-y| \le$$

$$\le C \max_{0 \le i \le n} (|x-x_0| + |x_0 - x_1| + \dots + |x_{i-1} - x_i|)^k |x-y| \le C^{k+1}|x-y|^{k+1}.$$

So, we get:

$$\|m^k(\Lambda e_1;0) - m^k(\Lambda e_1 - b;b)\|_{M^\infty_{k+1}(\Sigma)} \le$$

$$\le \sum_{1 \le i,j \le k+1} |x_i y_j| \cdot \|m^k(v^i_{0j};0) - m^k(0;v^i_{0j})\|_{M^\infty_{k+1}(\Sigma)} +$$

$$+ \sum_{1 \le i,j,p \le k+1} |x_i z_{jp}| \cdot \|m^k(0;v^i_{0j}) - m^k(v^i_{0j} - v^i_{1p};v^i_{1p})\|_{M^\infty_{k+1}(\Sigma)} + \dots$$

$$\dots + \sum_{1 \le i,j \le k+1} |x_i y_j| \cdot \|m^k(0;v^i_{Lj}) - m^k(v^i_{Lj} - b;b)\|_{M^\infty_{k+1}(\Sigma)} \le$$

$$\le [5(k+1)]^{k+1} \{ \sum_{1 \le i,j \le k+1} |x_i y_j| \cdot |v^i_{0j}|^{k+1} + \sum_{1 \le i,j,p \le k+1} |x_i z_{jp}| \cdot |v^i_{0j} - v^i_{1p}|^{k+1} +$$

$$+ \sum_{1 \le i,j,p \le k+1} |x_i z_{pj}| \cdot |v^i_{1p} - v^i_{2j}|^{k+1} + \dots + \sum_{1 \le i,j,p \le k+1} |x_i z_{pj}| \cdot |v^i_{L-1,p} - v^i_{Lj}|^{k+1} +$$

$$+ \sum |x_i y_j| \cdot |v^i_{Lj} - b|^{k+1} \} \le$$

(taking into account that $\frac{\Lambda}{\gamma}$ is assumed to be large enough and we chose $\alpha$ close to 1 and $\rho$ much smaller than $\gamma$)

$$\leq [5(k+1)]^{k+1}\{(k+1)^2 C\left(\frac{\Lambda}{\gamma}\right)^k\left(\left(\frac{\gamma}{k+1}\right)^2+\rho^2\right)^{\frac{k+1}{2}}+$$

$$+\sum_{s=0}^{L-1}(k+1)^3 C\left(\frac{\Lambda}{\gamma}\right)^k(2\rho\alpha^s)^{k+1}+(k+1)^2 C\left(\frac{\Lambda}{\gamma}\right)^k\left(\left(\frac{\gamma}{k+1}\right)^2+\rho^2\right)^{\frac{k+1}{2}}\}\leq$$

$$\leq A(k)\left[\Lambda^k\gamma+\Lambda^k\gamma\left(\frac{\rho}{\gamma}\right)^{k+1}\sum_{s=0}^{L-1}\alpha^{s(k+1)}\right]\leq$$

$$\leq A(k)\Lambda^k\gamma\left[1+\left(\frac{\rho}{\gamma}\right)^{k+1}\cdot\frac{1}{1-\alpha^{k+1}}\right].$$

Now choose $0<\alpha<1$, $\alpha$ close to 1, and choose $\frac{\rho}{\gamma}$ such, that

$$\left(\frac{\rho}{\gamma}\right)^{k+1}\cdot\frac{1}{1-\alpha^{k+1}}=\frac{1}{2}.$$

Certainly, $\frac{\rho}{\gamma}$ will be very small. So, we obtain (assuming that $\frac{\Lambda}{\gamma}$ is large enough)

$$\|m^k(\Lambda e_1;0)-m^k(\Lambda e_1-b;b)\|_{M^\infty_{k+1}(\Sigma)}\leq$$

$$\leq 2A(k)\Lambda^k\gamma=2A(k)\|\Lambda e_1\|^k|b|\frac{k+1}{k+2}\leq C(k)\int_{[0b]}|\Lambda e_1-w|^k d|w|.$$

### 3.4. Construction of a rapidly growing smooth function on a neighborhood of $L(\alpha,\gamma,\rho)$.

Now we define a special function on a neighborhood of $L(\alpha,\gamma,\rho)$, it will be a function "of the fastest growth" in a small neighborhood of $L(\alpha,\gamma,\rho)$, where its $k$-th derivatives will be bounded by 1.

We shall construct this function on a neighborhood of $L(\alpha)$ and then define it on a neighborhood of $L_i(\alpha,\gamma,\rho)$ and on a neighborhood of $L(\alpha,\gamma,\rho)$. $L(\alpha)$ is a union of the sets

$$W(0)=V(0)\bigcup(\bigcup_{j=1}^{k+1}H(0,j)),$$

$$W(s)=V(s)\bigcup(\bigcup_{j=1}^{k+1}H(s,j)\ (0<s<L)$$

$$W(L)=V(L)\bigcup(\bigcup_{j=1}^{k+1}H(L,j))\bigcup\{(x,\frac{(-\alpha)^L}{k+1}):0\leq x\leq\varepsilon\alpha^L\}\bigcup\{(\varepsilon\alpha^L,y):-1\leq y\leq 0\}$$

$$W(s_1)\bigcap W(s_2)=\emptyset \text{ for } |s_1-s_2|\geq 2$$

$$W(s)\bigcap W(s+1)=\{(-\varepsilon\alpha^{s+1},(-\alpha)^s)\} \text{ for } s=0,1,\dots,L-1$$

Now we begin to construct the function $\Psi(x,y)$ in a very thin neighborhood of $L(\alpha)$. "Very thin" means that we choose rectangles around every vertical or horizontal segment in such a way that the rectangles surrounding two noncoinciding parallel segments are disjoint.

Consider a smooth function $\phi(t)$ on $\mathbb{R}$, such that:
$\phi$ is identically zero on $(-\infty,0]$,
$\phi$ is identically constant $c$ on $[1,\infty)$,
$|\phi^{(k)}(t)|\leq 1$.

Consider an elementary labyrinth $L(\alpha)$. For $1\leq s\leq L-1$ consider the sets: $U_-(s)$ and $U_+(s)$ :

$$U_-(s)=\bigcup_{t<s}W(t),$$

$$U_+(s)=\bigcup_{t>s}W(t).$$

Note that

$$U_-(s)\bigcap W(s)=\{(-\varepsilon\alpha^{s-1},(-\alpha)^s)\},$$

$$W(s)\bigcap U_+(s)=\{(-\varepsilon\alpha^{s},(-\alpha)^{s+1})\}.$$

Let us define special functions $\Psi_s$ $(1\leq s\leq L-1)$ on $L(\alpha)$ :

$$\Psi_s(x,y)=\begin{cases}0, & (x,y)\in U_-(s),\\ \alpha^{sk}\phi(1-(-\alpha)^{-s}y), & (x,y)\in W(s),\\ \alpha^{sk}c, & (x,y)\in U_+(s).\end{cases}$$

One can easily check that this function may be naturally extended to a smooth function on a thin neighborhood of $L(\alpha)$ such that this extended function locally depends only upon $y$. (One needs only to check that the function $\alpha^{sk}\phi(1-(-\alpha)^{-s}y)$ and all its derivatives vanish at $y=(-\alpha)^s$, and this function is constant $\alpha^{sk}C$ near $y=(-\alpha)^{s+1}$). Note that

$$|\left(\frac{\partial}{\partial y}\right)^l\Psi_s|\leq h_l\alpha^{s(k-l)}\leq h_l,$$

for $1\leq l\leq k$, $h_k=1$, all mixed partial derivatives of $\Psi_s$ are identically zero.

Consider the function

$$\Psi(x,y)=\sum_{s=1}^{L-1}\Psi_s(x,y)$$

defined in a thin neighborhood of the elementary labyrinth $L(\alpha)$. Note that the supports of $\left(\frac{\partial}{\partial y}\right)^l\Psi_s$ $(1\leq l\leq k)$, are pairwise disjoint, so

$$|\left(\frac{\partial}{\partial y}\right)^l\Psi|\leq h_l,\ 1\leq l\leq k$$

(mixed partial derivatives are also identically zero on the neighborhood of $L(\alpha)$).

Now let us calculate

$$\Psi(0,1) - \Psi(\varepsilon\alpha^L, 0) = \sum_{s=1}^{L-1} [\Psi_s(0,1) - \Psi_s(\varepsilon\alpha^L, 0)]$$

The point $(0,1)$ belongs to $U_-(s)$ for all $s > 1$, the point $(\varepsilon\alpha^L, 0)$ belongs $U_+(s)$ for all $s < L$, so we get

$$\Psi(0,1) - \Psi(\varepsilon\alpha^L, 0) = \sum_{s=1}^{L-1} [\Psi_s(0,1) - \Psi_s(-\varepsilon\alpha^L, 0)] =$$

$$= \sum_{s=1}^{L-1} [0 - \alpha^{sk} c] = -c\alpha^k \cdot \frac{1 - \alpha^{(L-1)k}}{1 - \alpha^k} > -0.5c(1 - \alpha^k)^{-1}$$

(we assume that $\alpha^k(1 - \alpha^{(L-1)k}) > \frac{1}{2}$).

So, we have constructed a function defined in a narrow tube neighborhood of $L(\alpha)$, such that it is identically zero near the initial point of the elementary labyrinth, it is greater than $0.5\,c(1 - \alpha^k)^{-1}$ near the final point of the elementary labyrinth, locally it depends only upon the $y$-coordinate, its $l$-th derivatives are bounded by $h_l$

If we deal with a standard labyrinth $L(\alpha, \gamma, 1)$, we define a function $\tilde{\Psi}_{\alpha,\gamma}$ on the whole labyrinth by defining it on each elementary labyrinth separately just in the manner described above. The fact, that all such functions may be glued together, is obvious, - just put $\tilde{\Psi}_{\alpha,\gamma} = 0$ in a neighborhood of $\{(x, 1) : 0 \le x \le \gamma\}$ and $\{(0, y) : 0 \le y \le 1\}$ and put $\tilde{\Psi}_{\alpha,\gamma} = \Psi(\varepsilon\alpha^L, 0)$ in a neighborhood of $\{(x, -1) : 0 \le x \le \gamma\frac{k+2}{k+1}\}$ and $\{(\gamma\frac{k+2}{k+1}, y) : 0 \ge y \ge -1\}$.

Next, if we deal with a standard labyrinth $L(\alpha, \gamma, \rho)$ we define the function $\Psi_{\alpha,\gamma,\rho}$ by the formula

$$\Psi_{\alpha,\gamma,\rho}(x, y) = \rho^k \tilde{\Psi}_{\alpha,\gamma}(x, \frac{y}{\rho}).$$

**3.5. Shifted labyrinths.**

We need labyrinths obtained from $L(\alpha, \gamma, \rho)$ by shifts and rotations.

Let $L(\alpha, \gamma, \rho; \tilde{a}, z)$ denote the labyrinth obtained as above under the following assumptions:

(i) the origin is at the point $\tilde{a}$,

(ii) the direction of the $x$-axis coincides with the direction of the vector $z$.

We assume that the labyrinth $L(\alpha, \gamma, \rho; \tilde{a}, z)$ is endowed with a function $\Psi_L$ defined in a narrow tube neighborhood of $L(\alpha, \gamma, \rho; \tilde{a}, z)$ such that

(1) $\Psi_L$ is zero near $\tilde{a}$;

(2) $\Psi_L$ has the $l$-th derivatives, bounded by $h_l$, $h_k = 1$, $1 \le l \le k$.

(3) $\Psi_L \ge 0.5\rho^k c(1 - \alpha^k)^{-1}$ near the sink point $\tilde{b}$ of the labyrinth $L(\alpha, \gamma, \rho; \tilde{a}, z)$ . $\Psi_L$ is constant near the sink.

**3.6. Construction of $\mathcal{L}_{a,b,N}(z)$.**

Given $a, b \in \mathbb{R}^2,\ z \in \mathbb{R}^2,\ N$, we want to construct a set $\mathcal{L}_{a,b,N}(z)$ such that

(i) $\mathcal{L}_{a,b,N}(z)$ is the closure of a domain with a real analytic boundary;

(ii) $\mathcal{L}_{a,b,N}(z) \subset \{t \in \mathbb{R}^2 : |t - \frac{a+b}{2}| \leq |\frac{a-b}{2}\}$;

(iii) $a, b \in \partial\mathcal{L}_{a,b,N}(z)$;

(iv) $\|m^k(z-a;a) - m^k(z-b;b)\|_{M^{\infty}_{k+1}(\mathcal{L}_{a,b,N}(z))} \leq C(k)\int_{[ab]} |z-w|^k d|w|$;

(v) there exists a function $\Psi_{a,b,N} \in C^{\infty}(\mathcal{L}_{a,b,N}(z))$ such that

(*) $\Psi_{a,b,N}$ is identically zero near $a$;

(**) $\Psi_{a,b,N}$ is identically constant $N$ near $b$;

(***) $\sup\limits_{\substack{x \in \mathcal{L}_{a,b,N}(z) \\ |\alpha| = l}} |\Psi^{(\alpha)}_{a,b,N}(x)| \leq h_l,\ 1 \leq l \leq k,\ h_k = 1$

$\mathcal{L}_{a,b,N}(z)$ is constructed as follows:

Consider

$$C = \left\{ t \in \mathbb{R}^2 : |t - \frac{a+b}{2}| \leq |\frac{a-b}{2}| \right\}.$$

Consider the straight line passing through $\frac{a+b}{2}$, parallel to the vector $z$. Choose $\gamma > 0$ such that $\frac{|z|}{\gamma}$ is large enough (for the estimate of Section 3.3). Construct a shifted labyrinth with the source

$$\tilde{a} = \frac{a+b}{2} - \frac{\gamma}{2|z|}\, z,$$

with the sink

$$\tilde{b} = \frac{a+b}{2} + \frac{\gamma}{|z|}\left(\frac{k+2}{k+1} - \frac{1}{2}\right) z,$$

with the unit vector $\frac{z}{|z|}$ as the unit vector of the axis.

The parameters $\rho, \alpha, \varepsilon, L$ are chosen as follows:

First choose $\alpha$ as a solution of the equation

$$\frac{(1-\alpha^{k+1})^{\frac{k}{k+1}}}{1-\alpha^k} = \frac{2^k N}{\gamma^k c} = \frac{2^k N \cdot 4^k}{|a-b|^k c}$$

($c = \phi(1)$). This equation has a solution $\alpha$, $0 \leq \alpha < 1$, for any large $N$, because the function in the left hand side tends to infinity as $\alpha \to 1^-$.

Next, choose

$$\rho = \frac{\gamma}{2}(1-\alpha^{k+1})^{\frac{1}{k+1}}.$$

Then

$$\frac{\rho}{\gamma} = \left[\frac{1}{2}(1-\alpha^{k+1})\right]^{\frac{1}{k+1}} \ll 1,$$

so for any $\Sigma$, containing the labyrinth

$$\|m^k(z-\tilde{a},\tilde{a}) - m^k(z-\tilde{b},\tilde{b})\|_{M^{\infty}_{k+1}(\Sigma)} \leq 2A(k)\int_{[\tilde{a}\tilde{b}]} |z-w|^k d|w|.$$

Choose an odd $L \in \mathbb{N}$ such that $\alpha^k(1-\alpha^{(L-1)k}) > \frac{1}{2}$ and choose a small $\varepsilon > 0$ as it was explained above. Then we get

$$\frac{\left(\frac{\rho}{\gamma}\right)^{k+1}}{1-\alpha^{k+1}} = \frac{\left\{\left[\frac{1}{2}(1-\alpha^{k+1})\right]^{\frac{1}{k+1}}\right\}^{k+1}}{1-\alpha^{k+1}} = \frac{1}{2} < 1;$$

$$\frac{\rho^k}{1-\alpha^k} = \frac{\gamma^k}{2^k} \cdot \frac{(1-\alpha^{k+1})^{\frac{k}{k+1}}}{1-\alpha^k} = \frac{\gamma^k}{2^k} \cdot \frac{2^k N}{\gamma^k c} = \frac{N}{c}.$$

So the parameters of the shifted labyrinth are completely defined.

Consider the related function $\Psi_L$, $\Psi_L$ is identically zero near $\tilde{a}$, it is greater than

$$\frac{\rho^k}{1-\alpha^k}c = \frac{N}{c} \cdot c = N$$

near the sink of the labyrinth, it is infinitely differentiable in a neighborhood of the labyrinth and its derivatives of order $k$ are bounded by 1.

Connect $a$ to $\tilde{a}$ and $b$ to the sink by "almost straight line" segments. Extend $\Psi_L$ to this segments by constants. Consider a domain with a real analytic boundary such that it contains the labyrinth and these new "almost straight line" segments (except of the points $a, b$), This domain must be a neighborhood of the labyrinth such that the function $\Psi_L$ is defined there. The closure of this domain is exactly the labyrinth $\mathcal{L}_{a,b,N}(z)$ and the function $\Psi_L$ is exactly the function $\Psi_{a,b,N}$ we need.

To check the remaining point (iv) we proceed as follows:

$$m^k(z-a;a) - m^k(z-b;b) = [m^k(z-a;a) - m^k(z-\tilde{a};\tilde{a})] +$$

$$+[m^k(z-\tilde{a};\tilde{a}) - m^k(z-\tilde{b};\tilde{b})] + [m^k(z-\tilde{b};\tilde{b}) - m^k(z-b;b)].$$

So

$$\|m^k(z-a;a) - m^k(z-b;b)\|_{M^\infty_{k+1}(\mathcal{L})} \leq$$

$$\leq \|m^k(z-a;a) - m^k(z-\tilde{a};\tilde{a})\|_{M^\infty_{k+1}(\mathcal{L})} +$$

$$+\|m^k(z-\tilde{a};\tilde{a}) - m^k(z-\tilde{b};\tilde{b})\|_{M^\infty_{k+1}(\mathcal{L})} +$$

$$+\|m^k(z-\tilde{b};\tilde{b}) - m^k(z-b;b)\|_{M^\infty_{k+1}(\mathcal{L})} \leq$$

$$\leq C(k)\int_{[a\tilde{a}]} |z-w|^k d|w| + C(k)\int_{[\tilde{a}\tilde{b}]} |z-w|^k d|w| + C(k)\int_{[\tilde{b}b]} |z-w|^k d|w| \leq$$

$$\leq C(k)\int_{[ab]} |z-w|^k d|w|.$$

and we see that the required set is constructed together with the required function.

## References


1. J. Bonèt, R.W. Braun, R. Meise, B.A. Taylor, *Whitney's extension theorem for nonquasianalytic classes of ultradifferentiable functions*, Studia Math. **99:2** (1991), 155–184.
2. Yu. Brudnyi, P.A. Shvartsman, *A linear extension operator for a space of smooth functions defined on a closed subset of* $\mathbb{R}^n$, (Russian), Dokl. Akad. Nauk SSSR **280:2** (1985), 48–51.
3. Yu. Brudnyi, P.A. Shvartsman, *Generalization of Whitney's extension theorem*, Intern. Math. Res. Notes **3** (1994), 129–139.
4. A.P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Proc.Symp.Pure Math. **IV** (1961), 33–49.
5. G. Glaeser, *Étude de quelques algèbres Tayloriennes*, J.Analyse Math. **6** (1958), 1–125.
6. V.M. Gol'dshtein, T.G. Latfullin, S.K. Vodop'yanov, *Criteria for extension of functions of the class* $L^1_2$ *from unbounded plain domains*, Siber. Math. J.(English translation) **20:2** (1979), 298–301.
7. V.M. Gol'dshtein, Yu.G. Reshetnyak, S.K. Vodop'yanov, *On geometric properties of functions with generalized first derivatives*, Russian Math. Surveys (English translation) **34:1** (1979), 19–74.
8. L. Hörmander, *The Analysis of Linear Partial Differential Operators, Vol.1*, Springer-Verlag, 1983.
9. P.W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Mathematica **147:1-2** (1981), 71–88.
10. A. Jonsson, H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Math. Reports, 1, vol. 2, 1984.
11. V.N. Konovalov, *Description of traces of some classes of functions of several variables*, Preprint #84.21 64 pages (Russian) (1984), Institute of Mathematics, Academy of Sciences of Ukraine.
12. B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press, 1967.
13. V.G. Maz'ja, *Sobolev Spaces*, Springer–Verlag, 1985.
14. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.
15. H. Whitney, *Functions differentiable on the boundaries of regions*, Annals of Mathematics **35:3** (1934), 482–485.
16. H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.


Department of Mathematics and Computer Science, University of Miami, Coral Gables, FL 33124, USA

*E-mail address*: zobin@math.miami.edu