WM-02-106

Noncommutative Gauge Theory without Lorentz Violation

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(Dated: June 4, 2002)

Abstract

The most popular noncommutative field theories are characterized by a matrix parameter $\theta^{\mu\nu}$ that violates Lorentz invariance. We consider the simplest algebra in which the θ-parameter is promoted to an operator and Lorentz invariance is preserved. This algebra arises through the contraction of a larger one for which explicit representations are already known. We formulate a star product and construct the gauge-invariant Lagrangian for Lorentz-conserving noncommutative QED. Three-photon vertices are absent in the theory, while a four-photon coupling exists and leads to a distinctive phenomenology.

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I. INTRODUCTION

Over the past few years, the thrust of "beyond the standard model" particle theory has undergone a fundamental shift, from exploration of extensions of the standard model in flat, four-dimensional spacetime to those that follow from modifications of the structure of spacetime itself. One such possibility is the existence of extra spatial dimensions with either large or infinite radii of compactification, an idea motivated by the desire to eliminate the hierarchy between the gravitational and the weak scale. Aside from the existence of extra dimensions themselves, the reduction in the fundamental scale in these scenarios opens the possibility that new phenomena arising in string theory may also become of experimental relevance. One fascinating possibility that has met considerable interest in the recent literature is that spacetime may become noncommutative at distance scales just below those currently accessible in experiments [1–13]. In the canonical version of noncommutative spacetime, the position four-vector x^μ is promoted to an operator satisfying the commutation relation

$$[\hat{x}^\mu \, , \, \hat{x}^\nu] = i \, \theta^{\mu\nu} \quad , \tag{1.1}$$

where $\theta^{\mu\nu}$ is a real, constant matrix of ordinary c-numbers. Precisely this situation is realized in string theory when open strings propagate in the presence of a constant background antisymmetric tensor field [14]. Keeping in mind that all scales in nature may not be far above the weak scale, it is not unreasonable to consider the possibility that Eq. (1.1) could lead to observable consequences.

Connecting Eq. (1.1) to experimental observables requires that one formulate quantum field theories on a noncommutative space [15–22]. While ordinary fields are functions of a commuting, classical position four-vector x^μ, the algebraic properties of the underlying noncommutative theory can be reproduced by replacing ordinary multiplication by a star product. For example, in the canonical case, one defines a mapping between functions of noncommuting coordinates $\hat{x}^\mu$ and functions of the c-number coordinates x^μ via the Fourier transform

$$\hat{f}(\hat{x}) = \frac{1}{(2\pi)^n} \int d^n k \, e^{-ik\hat{x}} \int d^n x \, e^{ikx} f(x) \quad . \tag{1.2}$$

The requirement that

$$\hat{f}\hat{g} = \widehat{f \star g} \quad , \tag{1.3}$$

i.e. that the functions $f(x)$ and $g(x)$ yield a representation of the algebra under star multiplication, allows one to define the star product. In the canonical case, one obtains the Moyal-Weyl result:

$$(f \star g)(x) = f(x) \, \exp[\frac{i}{2} \overleftarrow{\partial_\mu} \, \theta^{\mu\nu} \, \overrightarrow{\partial_\nu}] \, g(x) \ . \tag{1.4}$$

A field theory action can now be represented as a functional of fields that depend only on commuting spacetime coordinates

$$S = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x))_\star \ , \tag{1.5}$$

where the $\star$ subscript indicates that all multiplications between fields are defined by Eq. (1.4). This representation of the action is nothing more than the mapping of the operator trace

$$S = \mathrm{Tr} \ \hat{\mathcal{L}} \tag{1.6}$$

to the space of ordinary functions.

Formulating gauge theories on noncommutative spaces introduces additional complications. For example, the simplest formulation of noncommutative U(1) gauge theory (one that does not require working order by order in the parameter θ) is only consistent if matter fields have charges 0 or ± 1; adding additional states with other charges makes it impossible to define a covariant derivative [16]. While U(N) gauge theories follow with relatively little effort after promoting ordinary to star multiplication [17], SU(N) gauge theories cannot be constructed in such a straightforward manner. These problems have been surmounted by Jurčo, *et al.* [21], who have shown that it is possible to maintain gauge invariance and noncommutativity simultaneously by employing a nonlinear field redefinition that is determined order by order in an expansion in the parameter θ. This approach has allowed construction of the full noncommutative standard model [22, 23], without relying on awkward embeddings of the standard model gauge group.

The most notable phenomenological feature of canonical noncommutative field theories is the violation of Lorentz invariance following from Eq. (1.1) [2–7]. Both θ^{i0} and $\epsilon^{ijk}\theta_{jk}$ are fixed three-vectors that define preferred directions in a given Lorentz frame. Phenomena such as the diurnal variation of collider cross sections have been noted in studies of noncommutative QED [2, 3], even though some bounds [5–7] from low-energy tests of Lorentz invariance seems to suggest that effects at colliders are likely to be negligible. Such constraints have been shown to be even more significant in noncommutative QCD [7], and are likely to persist in more general canonical models.

One approach to this problem is to ignore it, on the grounds that (most of) the bounds in question are obtained in theories whose Lagrangians are known only at lowest order in θ. These theories have not been shown to be renormalizable, while the most dangerous effects are obtained only through loop corrections [6, 7]. On the other hand, in simple situations where both all-orders and lowest-order Lagrangians are known, the bounds on Lorentz violation from loop effects are even *stronger* in the full theory [6]. We take the position that low-energy tests of Lorentz invariance [24] are likely to present a generic impediment to formulating a noncommutative standard model that is based on the canonical relation Eq. (1.1) and that is also phenomenologically relevant. One alternative is to push the noncommutativity into extra dimensions [8–10], leaving the four ordinary space-time dimensions commutative and Lorentz invariant [8, 9]. This has added benefits, for example, in allowing one to formulate a simple noncommutative QED including matter fields with arbitrary charges, provided these fields are restricted to an orbifold fixed point [8]. A more challenging approach is to formulate noncommutative field theories that are free from Lorentz violating effects, *ab initio*. It is this approach we wish to explore in our present work.

In this paper we will consider a new class of noncommutative theories in which the parameter θ in Eq. (1.1) is promoted to an operator $\hat{\theta}^{\mu\nu}$ that is *not* constant, but transforms as a Lorentz tensor. In the next section, we show that this algebra can be interpreted as a contraction of a famous Lorentz-invariant algebra due to Snyder [25] for which explicit representations are known. By treating $\hat{\theta}$ as an unphysical parameter, we find the appropriate generalizations of the star product and operator trace for functions of both x^μ and $\theta^{\mu\nu}$. We then show how these results may be applied in constructing Lorentz-invariant Lagrangians for fields that are functions of x^μ alone. In the case of gauge theories, we accomplish this last step using the type of nonlinear field redefinitions introduced in the context of noncommutative SU(N) gauge theories [21]. As a concrete example, we formulate Lorentz-invariant noncommutative QED and show that 4-photon interactions are present, while vertices with an odd number of photons do not occur. In the fourth section we undertake a brief phenomenological investigation of light-by-light scattering in this theory, and in the final section we summarize our conclusions.

II. ALGEBRA AND STAR-PRODUCT

Let us consider the simplest generalization of Eq. (1.1) in which $\theta^{\mu\nu}$ is promoted to an operator $\hat{\theta}^{\mu\nu}$ in the same algebra as the coordinates:

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= i\hat{\theta}^{\mu\nu} \ , \\ \left[\hat{\theta}^{\mu\nu}, \hat{x}^\lambda\right] &= 0 \ , \\ \left[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\alpha\beta}\right] &= 0 \ . \end{aligned} \tag{2.1}$$

One could proceed immediately to discuss the algebra of functions $\hat{f}(\hat{x}, \hat{\theta})$, as well their mapping to ordinary functions $f(x, \theta)$ and the associated star product. However, it is useful first to display an explicit representation of the operators $\hat{x}$ and $\hat{\theta}$ that makes the Lorentz invariance of Eq. (2.1) manifest. We accomplish this by contracting another Lorentz-invariant algebra for which representations are already known.

Snyder proposed an algebra of noncommutative spacetime coordinates leading to a Lorentz-invariant discrete spacetime [25],

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= ia^2 \hat{M}^{\mu\nu} \ , \\ \left[\hat{M}^{\mu\nu}, \hat{x}^\lambda\right] &= i\left(\hat{x}^\mu g^{\nu\lambda} - \hat{x}^\nu g^{\mu\lambda}\right) \ , \\ \left[\hat{M}^{\mu\nu}, \hat{M}^{\alpha\beta}\right] &= i\left(\hat{M}^{\mu\beta} g^{\nu\alpha} + \hat{M}^{\nu\alpha} g^{\mu\beta} - \hat{M}^{\mu\alpha} g^{\nu\beta} - \hat{M}^{\nu\beta} g^{\mu\alpha}\right) \ , \end{aligned} \tag{2.2}$$

where $g^{\mu\nu} = diag\,(+,-,-,-)$. The last two commutation relations involving the $\hat{M}^{\mu\nu}$ are those of the generators of the Lorentz group, while the first is new [25, 26]. (Together they imply $\hat{M}$ and $\hat{x}/a$ can be identified as the generators of SO(4,1).) Snyder's representation of this algebra is obtained by considering a 5-dimensional space with coordinates $\eta_0, \ldots, \eta_4$ and metric $diag\,(+,-,-,-,-)$, on which ordinary Lorentz transformations act only on the first four coordinates. Let us define $\eta_\mu \equiv (\eta_0, \eta_1, \eta_2, \eta_3)$, $\eta^\mu \equiv (\eta_0, -\eta_1, -\eta_2, -\eta_3)$, and

$$\begin{aligned} \hat{x}^\mu &= ia\left(\eta_4 \frac{\partial}{\partial \eta_\mu} + \eta^\mu \frac{\partial}{\partial \eta_4}\right) \ , \\ \hat{M}^{\mu\nu} &= i\left(\eta^\mu \frac{\partial}{\partial \eta_\nu} - \eta^\nu \frac{\partial}{\partial \eta_\mu}\right) \ . \end{aligned} \tag{2.3}$$

Transformations that leave both η_4 and the quadratic form $\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2$ invariant are Lorentz transformations of the η_μ; such transformations induce ordinary Lorentz transformations on the coordinates $\hat{x}^\mu$. From Eq. (2.3), it is not hard to show that the spatial coordinate operators $\hat{x}^i$ do not have a continuous spectrum, but rather have eigenvalues that are integers times the length scale a. The time coordinate x^0, on the other hand, can be shown to have a continuous spectrum.

The contraction of an algebra is a simpler one obtained by taking the limit of some parameter. We consider the rescaling

$$\hat{M}^{\mu\nu} = \hat{\theta}^{\mu\nu}/b \ . \tag{2.4}$$

and the limit

$$b \to 0 \ , \ a \to 0 \ , \tag{2.5}$$

with the ratio of a^2 and b held fixed,

$$\frac{a^2}{b} \to 1 \ . \tag{2.6}$$

The result of this contraction is the set of commutation relations given in Eq. (2.1). Lorentz transformations in the operator algebra are generated by $\hat{M}^{\mu\nu}$, which has the following commutation relation with $\hat{\theta}^{\alpha\beta}$:

$$\left[\hat{M}^{\mu\nu}, \hat{\theta}^{\alpha\beta}\right] = i\left(\hat{\theta}^{\mu\beta}g^{\nu\alpha} + \hat{\theta}^{\nu\alpha}g^{\mu\beta} - \hat{\theta}^{\mu\alpha}g^{\nu\beta} - \hat{\theta}^{\nu\beta}g^{\mu\alpha}\right) \ . \tag{2.7}$$

This is sufficient to establish that $\hat{\theta}^{\mu\nu}$ transforms as a Lorentz tensor and that Eq. (2.1) is Lorentz covariant. One may also define a momentum operator whose commutation relations with $\hat{M}$ and $\hat{\theta}$ are identical to that of $\hat{x}$, but this will not be relevant to the subsequent discussion. Noting that $a \to 0$ is part of the limit, we see that the contracted algebra corresponds to a continuum limit of Snyder's quantized spacetime.

With $\hat{\theta}^{\mu\nu}$ as an additional fundamental operator, elements of the group defined locally by Eq. (2.1) depend on both $\hat{x}^\mu$ and $\hat{\theta}^{\mu\nu}$. Ordinary c-number functions can again be related to these elements through a Fourier transform, though in this case over an extended set of variables. If $\hat{f} = \hat{f}(\hat{x}, \hat{\theta})$ is a member of the operator algebra, we define a relation to ordinary functions $f(x, \theta)$ by

$$\hat{f} = \int (d\alpha)(dB)\, e^{-i(\alpha\hat{x} + B\hat{\theta})}\, \tilde{f}(\alpha, B) \ , \tag{2.8}$$

where $\tilde{f}$ is the Fourier transform

$$\tilde{f}(\alpha, B) = \int (dx)(d\theta)\, e^{i(\alpha x + B\theta)}\, f(x,\theta) \ . \tag{2.9}$$

In these equations, the measures of integrations are defined by $(d\alpha) \equiv (2\pi)^{-4} d^4\alpha$, $(dB) \equiv (2\pi)^{-6} d^6 B$, $(dx) \equiv d^4 x$ and $(d\theta) \equiv d^6\theta$; the $B_{\mu\nu}$ and $\theta^{\mu\nu}$ are antisymmetric parameters, and index contraction is implicit in the products $\alpha x = \alpha_\mu x^\mu$ and $B\theta \equiv B_{\mu\nu}\theta^{\mu\nu}/2$. The measure

$$d^6 B = dB_{12} dB_{23} dB_{31} dB_{01} dB_{02} dB_{03} \tag{2.10}$$

can be shown to be Lorentz invariant if $B_{\mu\nu}$ transforms like a second-rank Lorentz tensor. The x^μ are a set of ordinary commuting coordinates, and the $\theta^{\mu\nu}$ (no hat) are a set of new commuting parameters in ordinary function space that correspond to the $\hat{\theta}^{\mu\nu}$. While the operators $\hat{x}$ and $\hat{\theta}$ are related through commutation relations, the commuting parameters x and θ are completely independent of each other. (This reflects the degrees of freedom associated with the 10 linearly independent generators of SO(4,1).)

The mapping from the operator algebra to the space of ordinary functions allows one to define a star-product through the requirement Eq. (1.3). The derivation, as usual, begins with the product

$$\hat{f}\hat{g} = \int (d\alpha)(dB)(d\gamma)(d\Delta)\, e^{-i(\alpha\hat{x} + B\hat{\theta})}\, e^{-i(\gamma\hat{x} + \Delta\hat{\theta})}\, \tilde{f}(\alpha, B)\, \tilde{g}(\gamma, \Delta) \ , \tag{2.11}$$

which is then simplified using the Baker-Campbell-Hausdorff formula,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+\dots} \ . \tag{2.12}$$

As a consequence of Eq. (2.1), the expansion in Eq. (2.12) terminates after the first commutator and, after some manipulation, one obtains the same $\star$-product as in the canonical case except for the presence of the extra argument θ:

$$(f \star g)(x,\theta) = f(x,\theta)\, \exp[\frac{i}{2}\, \overleftarrow{\partial_\mu}\, \theta^{\mu\nu}\, \overrightarrow{\partial_\nu}]\, g(x,\theta) \ . \tag{2.13}$$

This star product is manifestly Lorentz covariant; the Lorentz transformation properties of θ are identical to those of $\hat{\theta}$, as one can show via the mapping defined in Eqs. (2.8) and (2.9).

We also require a generalization of the operator trace. As a trace is a mapping from an operator algebra to numbers that is linear, positive ($\mathrm{Tr}\, \hat{f}\hat{f}^\dagger \geq 0$), and cyclic ($\mathrm{Tr}\hat{f}\hat{g} = \mathrm{Tr}\hat{g}\hat{f}$), we propose

$$\mathrm{Tr}\hat{f} = \int d^4x\, d^6\theta\, W(\theta)\, f(x,\theta) \ . \tag{2.14}$$

The weighting function $W(\theta)$ will allow us to work with truncated power series expansions of functions in θ. Therefore, we assume that the weighting function is positive and for any large $|\theta^{\mu\nu}|$ falls to zero quickly enough so that all integrals are well defined. Moreover, we assume W is even in θ, so that

$$\int d^6\theta\, W(\theta)\, \theta^{\mu\nu} = 0 \ . \tag{2.15}$$

Field theory actions follow from Eq. (1.6),

$$S = \int d^4x\, d^6\theta\, W(\theta)\, \mathcal{L}(\phi, \partial\phi)_\star \ . \tag{2.16}$$

As $\mathcal{L}(\phi, \partial\phi)_\star$ depends in general on both x and θ, the object that takes the role the ordinary Lagrangian will be the θ-integrated quantity,

$$\mathcal{L}(x) = \int d^6\theta\, W(\theta)\, \mathcal{L}(\phi, \partial\phi)_\star \ . \tag{2.17}$$

III. GAUGE THEORY

The star product that we have formulated allows us to reproduce the noncommutativity of the operators $\hat{x}^\mu$ and $\hat{\theta}^{\mu\nu}$ while working instead with functions of the classical variables x^μ and $\theta^{\mu\nu}$. Ordinary quantum field theories involve fields that are functions of x alone, suggesting two possible ways to proceed. For a theory without gauge invariance, we may simply choose our fields $\phi(x,\theta)$ to be functions of x only

$$\phi(x,\theta) \equiv \phi(x) \ , \tag{3.1}$$

and construct an action using the trace described in the previous section. For example, the Lagrangian for ϕ^4 theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\int d^6\theta\, W(\theta)\, (\phi\star\phi)^2 \ . \tag{3.2}$$

Here we have used

$$\int d^4x\, f \star g = \int d^4x\, f\, g \ , \tag{3.3}$$

and the normalization condition

$$\int d^6\theta\, W(\theta) = 1 \ , \tag{3.4}$$

to simplify the result.

On the other hand, if the field ϕ transforms as some representation of a gauge group G, then it is no longer possible to choose ϕ to be a function of x only, as θ dependence is introduced via the noncommutative generalization of the gauge transformation. Consider a U(1) gauge theory with a matter field ψ and gauge field A. Under a gauge transformation parameterized by $\Lambda(x,\theta)$, the fields transform as

$$\psi(x,\theta) \to \psi'(x,\theta) = U \star \psi(x,\theta) \ \ , \tag{3.5}$$

$$A_\mu(x,\theta) \to A'_\mu(x,\theta) = U \star A_\mu(x,\theta) \star U^{-1} + \frac{i}{e} U \star \partial_\mu U^{-1} \ \ , \tag{3.6}$$

where

$$U = (e^{i\Lambda})_\star \ \ . \tag{3.7}$$

It is straightforward to confirm that the Lagrangian

$$\mathcal{L} = \int d^6\theta\, W(\theta) \left[-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i \not{D} - m) \star \psi \right] \tag{3.8}$$

is gauge invariant, provided that

$$D_\mu = \partial_\mu - ieA_\mu \ \ , \tag{3.9}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu \stackrel{\star}{,} A_\nu] \, . \tag{3.10}$$

Superficially, Eqs. (3.8–3.10) are the same as in the case of canonical noncommutative QED [16], aside from the fact that our construction of the trace averages over the parameter θ. One must keep in mind, however, that the fields in Eqs. (3.8–3.10) are functions of both x and θ, and cannot be identified with the ordinary quantum fields $\psi(x)$ and $A^\mu(x)$.

To proceed, we will expand the fields as a power series in the variable θ, and demonstrate that the coefficients, which are functions of x alone, can be expressed solely in terms of ordinary quantum fields. The nonlinear field redefinition is fixed by the constraints of noncommutativity and gauge invariance. This approach is largely the same as the one employed in the construction of SU(N) noncommutative gauge theories in Refs. [21, 22].

The expansion in θ in our case is valid given the presence of the weighting function $W(\theta)$ that renders the integral of higher order terms small. Let us demonstrate this approach by constructing the Lagrangian for the pure gauge sector of our U(1) theory.

We begin by expanding both the gauge parameter Λ and the gauge field A^μ

$$\Lambda_\alpha(x,\theta) = \alpha(x) + \theta^{\mu\nu}\Lambda^{(1)}_{\mu\nu}(x;\alpha) + \theta^{\mu\nu}\theta^{\eta\sigma}\Lambda^{(2)}_{\mu\nu\eta\sigma}(x;\alpha) + \cdots \ , \tag{3.11}$$

$$A_\rho(x,\theta) = A_\rho(x) + \theta^{\mu\nu}A^{(1)}_{\mu\nu\rho}(x) + \theta^{\mu\nu}\theta^{\eta\sigma}A^{(2)}_{\mu\nu\eta\sigma\rho}(x) + \cdots \ . \tag{3.12}$$

We identify the first term in each expansion as the ordinary, x-dependent gauge parameter and gauge field, respectively. In an Abelian gauge theory, two gauge transformations parameterized by $\alpha(x)$ and $\beta(x)$ satisfy the property that

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi(x) = 0 \ , \tag{3.13}$$

where ψ is a matter field transforming infinitesimally as

$$\delta_\alpha\psi(x) = i\,\alpha(x)\psi(x) \ . \tag{3.14}$$

In the noncommutative theory, we require that the field $\psi(x,\theta)$ satisfy

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi(x,\theta) = 0 \ , \tag{3.15}$$

where the infinitesimal transformation

$$\delta_\alpha\psi(x,\theta) = i\,\Lambda_\alpha(x,\theta) \star \psi(x,\theta) \tag{3.16}$$

follows from Eq. (3.5). Eq. (3.15) requires that the parameter Λ satisfy

$$i\delta_\alpha\Lambda_\beta - i\delta_\beta\Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = 0 \ , \tag{3.17}$$

from which the transformation properties of $\Lambda^{(1)}$ and $\Lambda^{(2)}$ in Eq. (3.11) may be deduced. It may then be shown [21] that the following functions of the ordinary gauge parameter and gauge field satisfy this consistency condition:

$$\Lambda^{(1)}_{\mu\nu}(x;\alpha) = \frac{e}{2}\partial_\mu\alpha(x)A_\nu(x) \ , \tag{3.18}$$

$$\Lambda^{(2)}_{\mu\nu\eta\sigma}(x;\alpha) = -\frac{e^2}{2}\partial_\mu\alpha(x)A_\eta(x)\partial_\sigma A_\nu(x) \ . \tag{3.19}$$

Similarly, the requirement that the noncommutative gauge field $A(x,\theta)$ transforms infinitesimally as

$$\delta_\alpha A_\sigma = \partial_\sigma \Lambda_\alpha + i[\Lambda_\alpha \stackrel{\star}{,} A_\sigma] \ , \tag{3.20}$$

which follows from Eq. (3.6), is sufficient to determine the correct transformation properties of $A^{(1)}$ and $A^{(2)}$. These are reproduced by

$$A^{(1)}_{\mu\nu\rho}(x) = -\frac{e}{2} A_\mu(\partial_\nu A_\rho + F^0_{\nu\rho}) \ , \tag{3.21}$$

$$A^{(2)}_{\mu\nu\eta\sigma\rho}(x) = \frac{e^2}{2}(A_\mu A_\eta \partial_\sigma F^0_{\nu\rho} - \partial_\nu A_\rho \partial_\eta A_\mu A_\sigma + A_\mu F^0_{\nu\eta} F^0_{\sigma\rho}) \ , \tag{3.22}$$

where $F^0_{\mu\nu}$ represents the ordinary Abelian field strength tensor

$$F^0_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{3.23}$$

We may now express the noncommutative field strength tensor in terms of the ordinary gauge field $A^\mu(x)$. We find

$$\begin{aligned} F_{\mu\nu} \ =& \ F^0_{\mu\nu} + e\,\theta^{\kappa\lambda}(F^0_{\mu\kappa}F^0_{\nu\lambda} - A_\kappa \partial_\lambda F^0_{\mu\nu}) + \frac{e^2}{2}\theta^{\kappa\lambda}\theta^{\rho\eta}[F^0_{\kappa\nu}F^0_{\lambda\rho}F^0_{\eta\mu} - F^0_{\kappa\mu}F^0_{\lambda\rho}F^0_{\eta\nu} \\ & + \ A_\kappa(2F^0_{\mu\rho}\partial_\lambda F^0_{\eta\nu} - 2F^0_{\nu\rho}\partial_\lambda F^0_{\eta\mu} + F^0_{\lambda\rho}\partial_\eta F^0_{\mu\nu} - \partial_\rho F^0_{\mu\nu}\partial_\lambda A_\eta) + A_\kappa A_\rho \partial_\eta \partial_\lambda F^0_{\mu\nu}]. \end{aligned} \tag{3.24}$$

Photon self-interactions may be isolated by substituting this result into Eq. (3.8) and integrating over θ. For any Lorentz-invariant weighting function $W(\theta) \equiv W(\theta_{\mu\nu}\theta^{\mu\nu})$,

$$\int d^6\theta\, W(\theta)\, \theta^{\mu\nu}\theta^{\eta\rho} = \frac{1}{12}\langle\theta^2\rangle(g^{\mu\eta}g^{\nu\rho} - g^{\mu\rho}g^{\eta\nu}) \ , \tag{3.25}$$

where we have defined the expectation value

$$\langle\theta^2\rangle \equiv \int d^6\theta\, W(\theta)\, \theta_{\mu\nu}\theta^{\mu\nu} \ . \tag{3.26}$$

Photon self-interaction terms that are odd in θ vanish under this integration, so that the lowest-order nonstandard vertex is given by:

$$\mathcal{L} = \frac{\pi\alpha}{12}\langle\theta^2\rangle[F^0_{\mu\nu}F^{0\nu\eta}F^0_{\eta\rho}F^{0\rho\mu} - (F^0_{\mu\nu}F^{0\mu\nu})^2] \ . \tag{3.27}$$

Notice that the $\langle\theta^2\rangle^{-1/4}$ is an energy scale that characterizes the onset of new physics; since one generally likes to keep this scale a free parameter in phenomenological studies, one need not specify anything more about the form of the weighting function $W(\theta)$, at least at the order to which we are working. It is interesting to note that Eq. (3.27) reduces to

$$\mathcal{L} = \frac{\pi\alpha}{6}\langle\theta^2\rangle[-(\mathbf{E}^2 - \mathbf{B}^2)^2 + 2(\mathbf{E}\cdot\mathbf{B})^2] \tag{3.28}$$

when expressed in terms of classical electric and magnetic fields, which differs in form from the famous Euler-Heisenberg low-energy effective Lagrangian following at the one-loop level in QED [27]

$$\mathcal{L}_{E-L} = \frac{2\alpha^2}{45 m_e^4}\,[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2]\ . \tag{3.29}$$

Eq. (3.29) is valid at energies small compared to the electron mass, while our expectation is that $\langle\theta^2\rangle^{-1/4}$ will be of order a TeV, based on the type of bounds that are typical in extensions of the standard model that modify the gauge sector. We therefore turn briefly to the high-energy collider physics of our scenario, where the effects of noncommutativity are more likely to be manifest.

IV. PHENOMENOLOGY

The vertices that follow from our Lorentz-invariant construction of noncommutative QED provide a rich hunting ground for the origins of new phenomena at colliders. Deviations in observable scattering cross sections follow from modifications to vertices that occur at tree level in the standard model, as well as from the existence of new vertices. Examples of the latter include direct two-photon-two-fermion couplings, as well as the four-photon interaction discussed in detail in the previous section. Here will focus on the scattering process $\gamma\gamma \to \gamma\gamma$, which has been discussed in the recent literature as a potential window on physics beyond the standard model [28, 29].

Given the labeling of momenta and Lorentz indices shown in Fig. 1, the interaction in Eq. (3.27) leads to the Feynman rule

$$\begin{aligned} V_{4\gamma} \;=\; \frac{1}{6} i e^2 \left\langle\theta^2\right\rangle \Big\{ & -4 p_1^{\mu_2} p_2^{\mu_1} p_3^{\mu_4} p_4^{\mu_3} + p_1^{\mu_2} p_2^{\mu_3} p_3^{\mu_4} p_4^{\mu_1} + p_1^{\mu_2} p_2^{\mu_4} p_3^{\mu_1} p_4^{\mu_3} \\ & -4 p_1^{\mu_3} p_2^{\mu_4} p_3^{\mu_1} p_4^{\mu_2} + p_1^{\mu_3} p_2^{\mu_1} p_3^{\mu_4} p_4^{\mu_2} + p_1^{\mu_3} p_2^{\mu_4} p_3^{\mu_2} p_4^{\mu_1} \\ & -4 p_1^{\mu_4} p_2^{\mu_3} p_3^{\mu_2} p_4^{\mu_1} + p_1^{\mu_4} p_2^{\mu_1} p_3^{\mu_2} p_4^{\mu_3} + p_1^{\mu_4} p_2^{\mu_3} p_3^{\mu_1} p_4^{\mu_2} \\ + \; \Big(g^{\mu_1\mu_2} \Big[& \,(p_1^{\mu_3} p_2^{\mu_4} + p_2^{\mu_3} p_1^{\mu_4})\, p_3\cdot p_4 + 4 p_4^{\mu_3} p_3^{\mu_4} p_1\cdot p_2 \end{aligned}$$

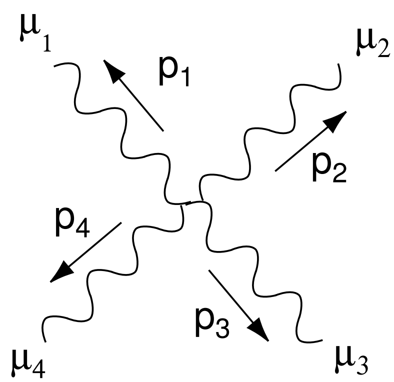

FIG. 1: Four-photon vertex.

$$
\begin{aligned}
& -p_4^{\mu_3} p_1^{\mu_4} p_2 \cdot p_3 - p_4^{\mu_3} p_2^{\mu_4} p_1 \cdot p_3 - p_1^{\mu_3} p_3^{\mu_4} p_2 \cdot p_4 - p_2^{\mu_3} p_3^{\mu_4} p_1 \cdot p_4 \Big] \\
& + [\ (12)(34) \to (34)(12)\ ] \\
& + [\ (12)(34) \to (13)(42)\ ] + [\ (12)(34) \to (42)(13)\ ] \\
& + [\ (12)(34) \to (14)(23)\ ] + [\ (12)(34) \to (23)(14)\ ] \Big) \\
+ \ & \Big(g^{\mu_1\mu_2} g^{\mu_3\mu_4} \Big[-4 p_1 \cdot p_2 \ p_3 \cdot p_4 + p_1 \cdot p_4 \ p_2 \cdot p_3 + p_1 \cdot p_3 \ p_2 \cdot p_4 \Big] \\
& + [\ (12)(34) \to (13)(42)\ ] + [\ (12)(34) \to (14)(23)\ ] \Big) \Big\} \ . \qquad (4.1)
\end{aligned}
$$

Placing the photons on shell, we may compute the differential scattering cross section in the photon center-of-mass frame. The noncommutative amplitude is 90° out of phase with the leading logarithmic contributions to the standard model background (see below), so we may write

$$\sigma \approx \sigma_{\rm NC} + \sigma_{\rm SM} \ . \qquad (4.2)$$

For unpolarized beams, we find

$$\frac{d\sigma_{\rm NC}}{d\cos\Theta^*} = \frac{19\pi}{128} \left(\frac{\langle \theta^2 \rangle}{12} \right)^2 \alpha^2 s^3 (3 + \cos^2\Theta^*)^2 \ , \qquad (4.3)$$

where $\sqrt{s}$ and Θ^* are the center of mass energy and scattering angle, respectively. It then follows that the noncommutative contribution to the total cross section ($0° < \Theta^* < 180°$) is given by

$$\sigma_{\rm NC} = \frac{133\pi}{80} \alpha^2 s^3 \left(\frac{\langle \theta^2 \rangle}{12} \right)^2 \ . \qquad (4.4)$$

To compare our result to the expectation in the standard model, we use the amplitudes given in Ref. [28] for light-by-light scattering in the high-energy limit s, $|t|$, $|u| \gg m_W^2$. So

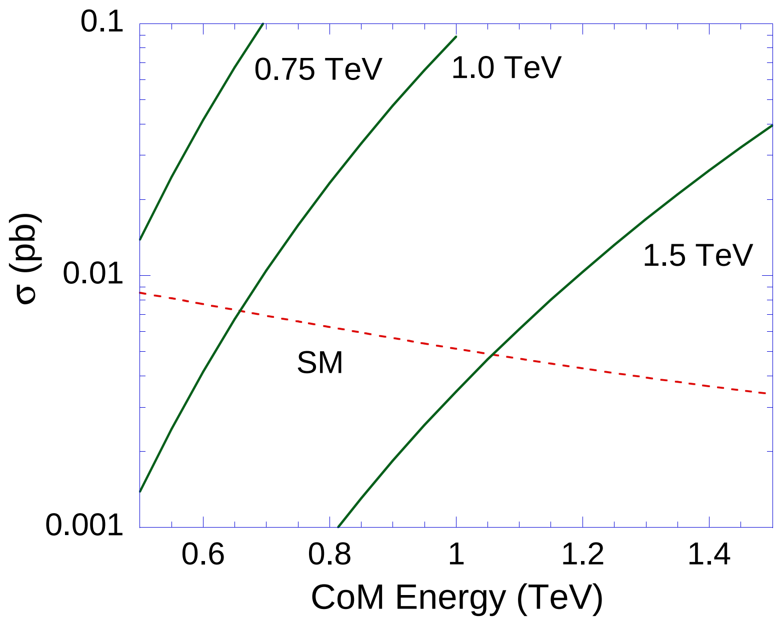

FIG. 2: Total cross sections $\sigma_{\rm NC}$ and $\sigma_{\rm SM}$ for $30^\circ < \Theta^* < 150^\circ$. Noncommutative results are labeled by the value of $\Lambda_{\rm NC}$, defined in the text.

that our discussion is self-contained, we reproduce the relevant results. The differential cross section is given by

$$\left(\frac{d\sigma}{d\cos\Theta^*}\right)_{SM} = \frac{1}{128\pi s}\left[(\mathrm{Im}\ F_{++++})^2 + (\mathrm{Im}\ F_{+-+-})^2 + (\mathrm{Im}\ F_{+--+})^2\right] \ , \tag{4.5}$$

where the dominant helicity amplitudes are mostly imaginary and

$$\mathrm{Im}\ F_{++++} = -16\pi\alpha^2 \left[\frac{s}{u}\ln\left|\frac{u}{m_W^2}\right| + \frac{s}{t}\ln\left|\frac{t}{m_W^2}\right|\right] \ , \tag{4.6}$$

$$\mathrm{Im}\ F_{+-+-} = -12\pi\alpha^2\frac{s-t}{u} - 16\pi\alpha^2\left[\frac{u}{s}\ln\left|\frac{u}{m_W^2}\right| + \frac{u^2}{st}\ln\left|\frac{t}{m_W^2}\right|\right] \ , \tag{4.7}$$

with

$$\mathrm{Im}\ F_{+--+}(s,t,u) = \mathrm{Im}\ F_{+-+-}(s,u,t) \ . \tag{4.8}$$

Figs. 2 and 3 show the comparison between our noncommutative result and the expectation in the standard model. Since the scale of new physics $\Lambda_{\rm NC}$ is characterized by a root-mean-square average of the components of $\theta^{\mu\nu}$, we define

$$\Lambda_{NC} = \left(\frac{12}{\langle\theta^2\rangle}\right)^{1/4} \ , \tag{4.9}$$

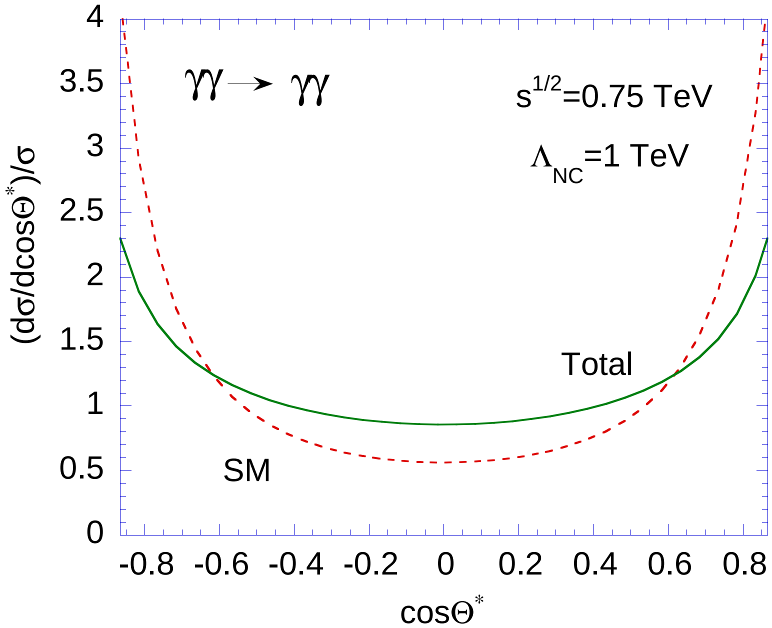

FIG. 3: Differential cross sections for $\sqrt{s} = 0.75$ TeV and $\Lambda_{\rm NC} = 1.0$ TeV, normalized to $\sigma(30^\circ < \Theta^* < 150^\circ)$. The dashed line indicates the standard model background and the solid line indicates the result when both the standard model and Lorentz-invariant NCQED interactions are present.

which also is a natural choice given Eq. (3.25). Note that the effective expansion parameter in the scattering amplitude is $s^2\langle\theta^2\rangle/12 \equiv s^2/\Lambda_{\rm NC}^4$, and each curve in Fig. 2 falls within a range where this ratio is less than one. The reader may easily estimate the size of higher-order corrections at any point in Fig. 2 by computing $s^2/\Lambda_{\rm NC}^4$. While the total cross section rises as s^3, which one would expect generically given the presence of new, effective contact interactions, the angular distribution is less forward and backward peaked in comparison to the standard model result. From the effective field theory point of view, any new physics can be parameterized by gauge-invariant interactions of the form $c_1 F_{\mu\nu}F^{\nu\eta}F_{\eta\rho}F^{\rho\mu} + c_2(F_{\mu\nu}F^{\mu\nu})^2$, for some coefficients c_1 and c_2. (Other possible interactions involving derivatives are irrelevant for a process in which all the photons are on shell.) While the scaling of the cross section with energy follows simply from dimensional analysis, the precise form of the dependence on scattering angle depends on the relative values of these coefficients. Note that our plots are evaluated for and $30^\circ < \Theta^* < 150^\circ$, the same angular range adopted in Ref. [28], which eliminates events close to the beam direction. For this choice, there are points in Fig. 2 where the noncommutative cross section substantially exceeds the standard model result, higher-order corrections in θ are under control, and our initial kinematical assumptions are satisfied. In a more complete phenomenological study, one would take into account the energy distribution of the initial photons, which are not monochromatic when produced via laser backscattering at an e^+e^- linear collider like CLIC or the NLC. Moreover, one can extract additional information from the polarized cross section since the polarization of the incident photon beams can be controlled to a large extent by the polarization of the lepton beams. We hope it is clear from the present example that our scenario may lead to potentially distinctive collider signals, and defer a complete investigation of these phenomenological issues to future work.

V. CONCLUSIONS

We have formulated a new class of noncommutative field theories in which the coordinate commutation relations are Lorentz covariant:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\hat{\theta}^{\mu\nu} \quad . \tag{5.1}$$

Here the parameter $\theta^{\mu\nu}$ of canonical noncommutative theories has been promoted to an operator $\hat{\theta}^{\mu\nu}$ that transform like a Lorentz tensor and all other relevant commutators are vanishing. We showed how Eq. (5.1) may be realized through the contraction of a larger Lorentz-invariant algebra for which explicit representations are already known.

Functions in the algebra of Eq. (5.1) depend on both $\hat{x}$ and $\hat{\theta}$. We may map these to functions of commuting variables provided the rule for multiplication is modified. We presented the star product of functions $f(x,\theta)$ that mimics the multiplication of operator functions $\hat{f}(\hat{x},\hat{\theta})$. By necessity, the commuting functions may depend not only on the familiar commuting variables x^μ, but also on a new set $\theta^{\mu\nu}$, that we treat as unphysical parameters; the operator trace may be expressed as an integral over both x and θ. With a star product and trace at hand, we showed how to formulate field theories in terms of functions of x^μ alone, and how to maintain gauge invariance through nonlinear field redefinitions.

We applied our formalism in constructing a Lorentz-invariant version of noncommutative QED. New vertices are present in this theory that are not found in ordinary QED, including two-fermion-two-photon and four-photon interactions, to name a few. However, unlike canonical noncommutative QED, no three-photon vertex is present. As an example of what might be observed experimentally if Lorentz-invariant noncommutative QED describes nature, we considered photon-photon elastic scattering at high energies, and obtained contributions that are significant with respect to the standard model background. The new noncommutative amplitude is present at tree level and at lower order in e^2 than the one-loop standard model result. The scattering cross section was shown to differ in both its energy dependence and angular distribution. At a photon-photon collider with $\sqrt{s} = 500$ GeV and an annual integrated luminosity of 100 fb^{-1}, one expects thousands of standard model events, while for $\Lambda_{\rm NC} = 0.75$ TeV the noncommutative effects can yield O(100%) corrections. Our results suggests that there is a clear opportunity at colliders to see the effects of Lorentz-conserving noncommutative QED if the noncommutativity scale is on the order of a TeV.

Acknowledgments

C.E.C. and C.D.C thank the National Science Foundation for support under Grant No. PHY-9900657. In addition, C.D.C. thanks the Jeffress Memorial Trust for support under Grant No. J-532.

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