Modification to one of the Charlie Problems

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Purpose The purpose of this article is to show why one of the conjectures given on the Open Problems List fails, and to give a possible reformulation of the problem that may be true.

Introduction We fix a family of norms $\{|\cdot|_x\}_{x\in\mathbb{R}^2}$ on the 2-jet space, that is they are norms on polynomials of degree 2 over $\mathbb{R}^2$. We assume the norms satisfy the so-called "Bounded Distortion", and "Approximate Translation Invariance" properties. These norms generate a norm on $C^2(\mathbb{R}^2)$ in the usual manner; we denote this norm by $||\cdot||_{C^2(\mathbb{R}^2)}$. The conjecture on the open problem list is the following:

Conjecture 0.1. *Let $I = \{(x,0) : -1.5 \leq x \leq 1.5\} \subset \mathbb{R}^2$, and $Q = [-1,1]^2$. Also suppose that we are given a set of polynomials $\{P^z\}_{z\in I}$ satisfying $|P^z|_z \leq 1$, $\forall z \in I$. Also suppose that there exists $F_0 \in C^2(Q)$ such that $J_z(F_0) = P^z$, $\forall z \in I$. Then given $\epsilon > 0$, there exists $F \in C^2(Q)$ satisfying:*

1. $J_z(F) = P^z$, $\forall z \in I$.
2. $||F||_{C^2(Q)} \leq 1 + \epsilon$.

The conjecture is false just because it fails along every vertical line. Pick the 2-jet space norm which is the maximum of all the partial derivatives of order ≤ 2. Parametrize $\mathbb{R}^2$ by (x, y). Suppose for example that $P^{(0,0)} = 1 + y$. Assume that there exists an extension of $\{P^z\}_{z\in I}$ to all of Q with norm ≤ 2; call this extension F. We know that $|\partial_{yy}F|$ is bounded by 2 on the y-axis. Therefore since $\partial_y F(0, 0) = 1$, we see that $\partial_y F(0, t) \geq 0.5$, $\forall t \in [-0.25, 0.25]$ (From the mean value theorem). By similar reasoning we see that $F(0, 0.25) \geq 1.125$, and therefore any extension has norm larger than some fixed constant, even though $|P^z|_z \leq 1$, $\forall z \in I$.

Remark 0.2. *The reason that the conjecture is false is because the family of norms are not "natural", to be made precise in the following sense:*

Definition 0.3. *A family of norms $\{|\cdot|_x\}_{x\in\mathbb{R}^n}$ on the m-jet space is "natural" if the following holds $\forall x \in \mathbb{R}^n$, and for all polynomials P of degree $\leq m$:*

$$|P|_x = \inf\{||F||_{C^m(\mathbb{R}^n)} : J_xF = P\} \tag{1}$$

The following is due to Charlie Fefferman:

Given a family of norms on the m-jet space: $\{|\cdot|_x\}_{x\in\mathbb{R}^n}$ satisfying the Bounded Distortion and Approximate Translation Invariance properties. Then we may define a new family of norms as follows:

$$|P|'_x = \inf\{||F||_{C^m(\mathbb{R}^n)} : J_xF = P\},\ \forall x \in \mathbb{R}^n \tag{2}$$

Remark 0.4. *The m-jet space norms defined by* (2) *are natural. The reasoning goes as follows:*

Given a Polynomial P of degree $\leq m$, and given $x \in \mathbb{R}^n$, we must show that:

$$|P|'_x = \inf\{||F||'_{C^m(\mathbb{R}^n)} : J_x F = P\}, \forall x \in \mathbb{R}^n. \tag{3}$$

Where $||F||'_{C^m(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |J_x F|'_x$.

We note the following:

$$||F||'_{C^m(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |J_x F|'_x \leq ||F||_{C^m(\mathbb{R}^n)}$$

The second inequality follows directly from (2)*.*
Now implied by the preceding remarks is the following

$$|P|'_x = \inf\{||F||_{C^m(\mathbb{R}^n)} : J_x F = P\} \geq \inf\{||F||'_{C^m(\mathbb{R}^n)} : J_x F = P\}$$

Note that this is one direction of the equality we wish to prove, and the other direction is trivial. Thus we have shown that the norms $\{|\cdot|'_x\}$ are "natural" m-jet norms.

With these remarks we can now state a reformulation of (0.1) that takes into account this "natural" condition:

Conjecture 0.5. *Let $I = \{(x, 0) : -1.5 \leq x \leq 1.5\} \subset \mathbb{R}^2$, and $Q = [-1, 1]^2$. Also suppose that we are given a set of polynomials $\{P^z\}_{z \in I}$ satisfying* $\inf\{||F||_{C^2(\mathbb{R}^2)} : J_z F = P^z\} \leq 1, \forall z \in I$. *Also suppose that there exists $F_0 \in C^2(Q)$ such that $J_z(F_0) = P^z$, $\forall z \in I$. Then given $\epsilon > 0$, there exists $F \in C^2(Q)$ satisfying:*

1. *$J_z(F) = P^z$, $\forall z \in I$.*
2. *$||F||_{C^2(Q)} \leq 1 + \epsilon$.*

Note that this conjecture is essentially stating that for this specific setup we have a finiteness number $k^{\#}_{\epsilon} = 1, \forall \epsilon > 0$. Also note that the following condition is probably too strong:

$$\inf\{||F||_{C^2(\mathbb{R}^2)} : J_z F = P^z\} \leq 1$$

And should be replaced with something like:

$$\inf\{||F||_{C^2([-2,2]^2)} : J_z F = P^z\} \leq 1$$