Open Problems (updated July 2009)

Second Workshop on Whitney Problems

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1. Edward Bierstone and Pierre Milman's Problems

Extension problems for geometric classes (E.g., subanalytic, semialgebraic or, more generally, o-minimal structures).

Problem BM1. Extension of subanalytic functions Is there a Whitney extension theorem for C^m subanalytic functions on a closed subanalytic subset X of $\mathbb{R}^n$? (A function is *subanalytic* if its graph is subanalytic.)

As evidence, there seems to be a subanalytic extension involving loss of differentiability, as in our paper [Inv. Math. 151 (2003), 329–352].

2. Characterization of "tame" subanalytic sets by the extension property There is a remarkable subclass of subanalytic sets, called *semicoherent*, characterized by the following theorem.

Theorem 1.1. *Let X denote a compact subanalytic subset of $\mathbb{R}^n$. Then the following conditions are equivalent:*

(1) *Composite function property. If $p : M \to \mathbb{R}^n$ is a proper real-analytic mapping with $p(M) = X$, then the ring of composite C^∞ functions $p^*C^\infty(\mathbb{R}^n)$ (where $p^*(g) := g \circ p$) is closed in $C^\infty(M)$.*

(2) *$C^\infty(X)$ is the intersection of all $C^m(X)$.*

(3) *Natural local algebraic invariants of X (e.g., the Hilbert-Samuel function) are upper-semicontinuous (in the subanalytic Zariski topology).*

(4) *For each l, the degree $\leq l$ part of the C^m paratangent bundle ($m \geq l$) stabilizes as m increases.*

(5) *X is semicoherent (i.e., satisfies a stratified version of the Oka-Cartan coherence theorem).*

(6) *There is a uniform bound for a local invariant of X called the Chevalley function (which compares algebraic and metric notions of order of vanishing).*

The various equivalences are proved in [Ann. of Math. 147 (1998), 731–785], [Duke Math. J. 83 (1996), 607–620], [Inv. Math., loc. cit.]. We show that, if X is semicoherent, then there is an extension operator $E : C^\infty(X) \to C^\infty(R^n)$. Thus we get estimates on the C^k seminorms of an extension: Given $k \in N$ and $K \subset \mathbb{R}^n$ compact, there exist $l = l(k, K) \in N$ and $L = L(k, K) \subset X$ compact, such that

$$\|E(f)\|_k^K \leq c\|f\|_l^L.$$

Problem BM2. Can the class of semicoherent sets be characterized by the extension property?

We prove:

Theorem 1.2. *Suppose there is an extension operator as above with an estimate $l(0, K) = 0$ on the zeroth seminorms, for every compact K. Then X is semicoherent.*

So the preceding question can be reformulated: *Does semicoherence imply the extension property with $l(0, K) = 0$?*

We do not know whether this is true in simple examples (e.g., X = union of the x-axis and the parabola $y = x^2$ in $\mathbb{R}^2$).

2. Problems of Yu. Brudnyi and N. Kalton

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of all bounded subsets of $\mathbb{R}^n$.

The local polynomial (best) approximation of order k is a function $E_k : \ell_\infty^{loc}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}_+$ given by

$$E_k(S\,; f) := \inf\big\{\|f - P\|_{\ell_\infty(S)}\,;\, P \in \mathcal{P}_{k-1,n}\big\}. \tag{1}$$

Notice that the order k differs by 1 from the corresponding degree of the approximating polynomials.

Also, *k-oscillation* of $f : \mathbb{R}^n \to \mathbb{R}$ on a set $S \subset \mathbb{R}^n$ is given by

$$\omega_k(S\,; f) := \sup\big\{|\Delta_h^k f(x)|\,;\; x + jh \in S, \quad j = 0, 1, \ldots, k\big\}, \tag{2}$$

where

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh).$$

Let S be a closed subset of $\mathbb{R}^n$. We define the *Whitney constant* $w_k(S)$ by

$$w_k(S) := \sup\big\{E_k(S\,; f)\,;\, f \in C(S) \quad \text{and} \quad \omega_k(S\,; f) \le 1\big\}. \tag{3}$$

We also define the *global Whitney constant* $w_k(n)$ by

$$w_k(n) := \sup\{w_k(S)\,;\, S \subset \mathbb{R}^n \quad \text{bounded and convex}\}. \tag{4}$$

In the spirit of the Whitney paper *On functions with bounded n-differences, J. Math. Pure Appl.* **9**, *No. 3 (1957), 67–95*, who considered the case of dimension one[1], let us consider also the constants $w_k^*(n)$ and $w_k^{**}(n)$ defined by (3) with $S := \mathbb{R}_+^n$ and $S := \mathbb{R}^n$. One can prove the following estimates:

$$w_k^*(n) \le 2, \quad w_k^{**}(n) \le \min_{1\le j\le n} 1 \Big/ \binom{n}{j}.$$

In contrast, the sharp upper bound for $w_k(n)$ depends on the dimension, and in fact, $\lim_{n\to\infty} w_k(n) = \infty$ if $k \ge 2$. We discuss this situation following the paper of Yu. Brudnyi and Kalton *Polynomial approximation on convex subsets of* $\mathbb{R}^n$*, Constr. Appr.* **16** *(2000), 161–199*.

In one-dimensional case the sharp value of Whitney constant $w_k := \omega_k(1)$ is known only for $k = 1$ and 2 ($w_1 = w_2 = \frac{1}{2}$). Whitney, 1957, proved that $\frac{8}{15} \le w_3 \le \frac{7}{10}$ and $w_k < \infty$ for all k. It is conjectured by Sendov that $w_k \le 1$ for all k. Recently it was proved that $w_k < 2 + e^{-2}$ and the Sendov conjecture was proved for $k \le 7$.

[1] In this case $w_k(1) = w_k([0, 1])$.

Now we present several conjectures and results due to Yu. Brudnyi and Kalton.

There is a fairly precise estimate for $w_2(n)$, i.e.,

$$\frac{1}{2}\log_2\Big(\Big\lfloor\frac{n}{2}\Big\rfloor+1\Big)\le w_2(n)\le\frac{1}{2}\lfloor\log_2 n\rfloor+\frac{5}{4}.$$

Curiously enough, $w_2(n)$ is almost attained not for the unit n-simplex S^n as it may be thought, but for $S^n\oplus S^n\subset\mathbb{R}^{2n}$. Meanwhile for S^n the precise asymptotic is given by

$$\lim_{n\to\infty}\frac{w_2(S^n)}{\log_2 n}=\frac{1}{4}\,.$$

We will write $w_k(\ell_p^n)$ instead of $w_k(S)$ when S is the closed unit ball of ℓ_p^n. Then $w_2(\ell_1^n)\approx\log n$ while $w_2(\ell_p^n)$ with $1<p\le\infty$ is equivalent, up to a logarithmic factor, to $(p-1)^{-1}$ as $p\to 1$. In the important for applications case of the n-cube (i.e., $p=\infty$) this constant is bounded by 802. (It is conjectured that $w_2(\ell_\infty^n)\le 2$.)

Now let $w_k^{(sym)}(n)$ be defined as in (4) but for centrally symmetric convex bodies. Then for some numerical constants $c_1,c_2>0$,

$$c_1\sqrt{n}\le w_3^{(sym)}(n)\le c_2\sqrt{n}\log(n+1).$$

As in the linear approximation case, this result can be improved for $w_3(\ell_p^n)$. For example, $w_3(\ell_2^n)\approx\log(n+1)$ and

$$c_1\log(n+1)\le w_3(\ell_\infty^n)\le c_2\big(\log(n+1)\big)^2.$$

There are also a few estimates for $k\ge 4$. In particular,

$$w_k^{(sym)}(X)\le cn^{\frac{k}{2}-1}\log(n+1),$$

while

$$w_k(\ell_p^n)\le cn^{\frac{(k-3)}{2}}\log(n+1)$$

for $2\le p\le\infty$ and $w_k(\ell_1^n)\approx\log(n+1)$.

Conjectures. (a) If $k\ge 2$, then

$$w_k(n)\approx w_k^{(sym)}(n)\approx n^{\frac{k}{2}-1}\log(n+1)$$

as $n\to\infty$.

This is proved for $k=2$ while the upper estimate for $w_k^{(sym)}(n)$ is established for $k\ge 2$. As for the lower bound, we only have $w_k(n)\ge w_k^{(sym)}(n)\ge c\sqrt{n}$ for $k\ge 3$.

(b) If $k\ge 3$ and $1\le p<\infty$, then

$$w_k(\ell_p^n)\approx\log(n+1)$$

as $n\to\infty$.

The result is established for $p=1$ and all $k\ge 2$ and for $k=3$ and $2\le p<\infty$, while the lower bound is established for all $k\ge 3$. It is quite possible, that it is way off the mark when $k\ge 4$.

(c) $w_2(\ell_\infty^n)$ is "small", say, $w_2(\ell_\infty^n)\le 2$. The only known results are $w_2(\ell_\infty^1)=\frac{1}{2}$, and $w_2(\ell_\infty^2)=1$, and $w_2(\ell_\infty^n)\le 802$ for $n\ge 3$. If the conjecture held, then for every convex function f on an n-cube Q we would have the inequality

$$E_2(Q\,;f)\le\omega_2(Q\,;f).$$

(d) If X is an infinite-dimensional Banach space, then $w_3(X) = \infty$.

3. Problems of A. Brudnyi and Yu. Brudnyi

3.1. Let X be a "smoothness" Banach space continuously embedded into $C(\mathbb{R}^n)$, e.g., $C^{k,\omega}(\mathbb{R}^n)$, Sobolev space $W_p^k(\mathbb{R}^n)$, where $\frac{k}{n} > \frac{1}{p}$, the Zygmund space of functions whose second difference $|\Delta_h^2 f(x)| = O(\|h\|)$ etc. Then the trace space $X|_S$ is well defined for every $S \subset \mathbb{R}^n$. This space is Banach under the canonical trace norm.

Consider a class $\mathcal{S}$ of subsets in $\mathbb{R}^n$ Given an integer $N \geq 1$ and a subset $S \subset \mathcal{S}$ define a functional $f \mapsto \delta_N(f; S; X)$ on $C(S)$ by

$$\delta_N(f; S; X) := \sup_{\Sigma} \|f|_\Sigma\|_{X|_\Sigma}, \tag{5}$$

where Σ runs over all N-point subsets of S.

A space X has finite property with respect to the class $\mathcal{S}$ if for some integer $N \geq 1$, constant $C > 0$ and every $S \subset \mathcal{S}$ and $f \in C(S)$

$$\|f\|_{X|_S} \leq C\delta_N(f; S; X). \tag{6}$$

We conventionally assume that the left-hand side is ∞ if $f \notin X|_S$. The minimal N for which (6) holds for all $S \subset \mathcal{S}$ is said to be the *finiteness constant* of X with respect to the class of sets $\mathcal{S}$ and is denoted by $\mathcal{F}_{\mathcal{S}}(X)$.

3.2. k-modulus of continuity is the function on $\ell_\infty^{loc}(\mathbb{R}^n) \times (0, +\infty)$ with range in $\mathbb{R}_+ \cup \{+\infty\}$ given by

$$\omega_k(t\,; f) := \sup_{\|h\| \leq t} \left\|\Delta_h^k f\right\|_{\ell_\infty(\mathbb{R}^n)}. \tag{7}$$

Here $\ell_\infty^{loc}(\mathbb{R}^n)$ is the (Fréchet) space of locally bounded functions f on $\mathbb{R}^n$ equipped with the collection of seminorms $\left\{\sup_C |f|\right\}$, where C runs over the family of compact subsets of $\mathbb{R}^n$.

A function $\omega : (0, +\infty) \to \mathbb{R}_+$ belongs to the class Ω_k if it satisfies the conditions

(a) ω is nondecreasing, continuous and

$$\omega(0+) = 0;$$

(b) for all $0 < t \leq s$

$$\frac{\omega(s)}{s^k} \leq \frac{\omega(t)}{t^k}.$$

In the sequel the functions of Ω_k will be called *k-majorants.*

Let $\omega \in \Omega_k$. The homogeneous Lipschitz space $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ consists of locally bounded on $\mathbb{R}^n$ functions f satisfying

$$|f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{t>0} \frac{\omega_k(t\,; f)}{\omega(t)} < \infty. \tag{8}$$

We also define the Banach space $\Lambda^{k,\omega}(\mathbb{R}^n)$ of Lipschitz functions of order k by

$$|f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + |f|_{\Lambda^{k,\omega}(\mathbb{R}^n)}. \tag{9}$$

In particular, let $\omega(t) := t^\sigma,\ 0 < \sigma \le k$, and s be the largest integer less than σ. Then the following equalities hold:

(a) If σ is noninteger or $\sigma = k$, then

$$\Lambda^{k,\omega}(\mathbb{R}^n) = C^{s,\bar{\omega}}(\mathbb{R}^n), \tag{10}$$

where $\overline{\omega}(t) := t^{\sigma - s}$ and the corresponding norms are equivalent.

(b) If $\sigma < k$ is integer, then

$$\Lambda^{k,\omega}(\mathbb{R}^n) = C^s\Lambda(\mathbb{R}^n)\,(= B^\sigma_\infty(\mathbb{R}^n)). \tag{11}$$

Let us recall that the space on the right-hand side is defined by the norm

$$\|f\|_{B^\sigma_\infty(\mathbb{R}^n)} := \|f\|_{\ell_\infty(\mathbb{R}^n)} + \max_{|\alpha|=\sigma-1} \sup_{t>0} \frac{\omega_2(t\,; D^\alpha f)}{t}\,. \tag{12}$$

3.3. Let $X \subset \mathbb{R}^n$ be a subset. By $\mathcal{K}_S$ we denote the set of cubes in $\mathbb{R}^n$ centered at S and of ℓ_∞-radius at most $diam\, S$. We write $Q_r(x)$ for the cube of radius r and center x and denote by $X_r(x)$ the set $Q_r(x) \cap X$ for $x \in X$. By $\mathbb{R}[x_1, \dots, x_n]$ we denote the space of real polynomials on $\mathbb{R}^n$ and by $\mathcal{P}_k \subset \mathbb{R}[x_1, \dots, x_n]$ the subspace of polynomials of degree k. For $k = 0$ we let $\mathcal{P}_{k-1} := \{0\}$.

We say that X is Markov if there is a constant $C = C(X, n, k)$ such that for every polynomial $p \in \mathcal{P}_k$, $k \ge 1$, and every $Q_r(x) \in \mathcal{K}_X$

$$\sup_{X_r(x)} ||\nabla p|| \le \frac{C}{r} \sup_{X_r(x)} |p|. \tag{13}$$

(Here $||\cdot||$ is the Euclidean norm on $\mathbb{R}^n$.)

(1) Let $X_j \subset \mathbb{R}^{n_j}$ be closed s_j-sets, $n_j - 1 < s_j \le n_j$, $1 \le j \le m$ (i.e., for each j there are positive constants a_j, b_j such that for every cube $B \in \mathcal{K}_{X_j}$ of radius r,

$$a_j r^{s_j} \le \mathcal{H}_{s_j}(B \cap X_j) \le b_j r^{s_j};$$

here $\mathcal{H}_{s_j}$ is the Hausdorff s_j-measure on X_j).
Then $X := X_1 \times \cdots \times X_m \subset \mathbb{R}^n$, $n := n_1 + \cdots + n_m$, is a Markov set.
(2) If $X \subset \mathbb{R}^n$ is a compact Markov set and $\phi : U \to V \subset \mathbb{R}^n$ is a C^1 diffeomorphism defined on a neighbourhood U of X, then $\phi(X)$ is Markov.
(3) If $X_j \subset \mathbb{R}^n$, $1 \le j \le m$, are Markov, then $\cup_j X_j$ is Markov.
(4) Any Lipschitz submanifold of $\mathbb{R}^n$ of dimension $\le n - 1$ is not Markov.

If there exists a positive number $r_0 < diam\, X$ such that inequality (13) is valid for all $Q_r(x) \in \mathcal{K}_X$ with $r \le r_0$, then X is called locally Markov.

The class of locally Markov subsets of $\mathbb{R}^n$ is denoted by $Mar^{loc}(\mathbb{R}^n)$. Clearly if X is compact and locally Markov, then X is Markov.

A subset $X \subset \mathbb{R}^n$ is said to be Markov of weak type if for each $x \in X$ there exists a sequence $\{r_j(x)\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ converging to 0 such that inequality (13) with $r = r_j(x)$, $j \in \mathbb{N}$, is valid for every $k \ge 1$ with the constant $C = C(x)$.

The class of Markov subsets of weak type of $\mathbb{R}^n$ is denoted by $Mar^w(\mathbb{R}^n)$.

A union of locally Markov subsets of $\mathbb{R}^n$ is Markov of weak type. In particular, any open subset of $\mathbb{R}^n$ is Markov of weak type.

Problems.

(1) Prove that finiteness constant $\mathcal{F}_{Mar^{loc}(\mathbb{R}^n)}(\Lambda^{k,\omega}(\mathbb{R}^n)) = \binom{n+k-1}{n} + 1$. This problem relates to interesting geometric questions. As an example, we formulate one of them. Does there exist a Markov set in the plane such that every its three-point subset does not belong to a straight line?

(2) Evaluate $\mathcal{F}_{\mathcal{S}}(\Lambda^{k,\omega}(\mathbb{R}^n))$, where $\mathcal{S}$ is the class of d-sets, $0 < d \leq n$.

(3) Let S belong to the class of d-sets, $0 < d \leq n$. Prove that there exists a linear extension operator from $\Lambda^{k,\omega}(\mathbb{R}^n)|_S$ to $\Lambda^{k,\omega}(\mathbb{R}^n)$ whose norm and depth depend only on k, n and S.[2]

In case $n-1 < d \leq n$ the results follow from a more general theorem proved by Yu. Brudnyi and A. Brudnyi.

(4) Characterize trace space $\Lambda^{k,\omega}(\mathbb{R}^n)|_S$ for S being a Lipschitz submanifold of $\mathbb{R}^n$.

(5) Evaluate finiteness constant $\mathcal{F}_{Mar^{w}(\mathbb{R}^n)}(C^{k,\omega}(\mathbb{R}^n))$.

4. Charles Fefferman's Problems

Problem F1. Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite. Let $\epsilon > 0$. How many computer operations does it take to compute a function $F_\epsilon \in C^m(\mathbb{R}^n)$ such that $F_\epsilon|_E = f$, with the C^m-norm of F_ϵ at most ϵ percent more than the least value possible (inf)?

To "compute a function" F in $C^m(\mathbb{R}^n)$ from data means the following: We enter the data into a computer, which performs "one-time work", then signals that it is ready to accept "queries". We may then query the computer, by inputting points x in $\mathbb{R}^n$. The computer responds to each query x by performing a calculation (the "query work"), and then printing out the values at x of F and its derivatives up to order m. The resources used to compute F are: The number of computer operations used in the one-time work; The number of computer operations used in the query work; and The storage, i.e. the number of real numbers or integers that can be stored in RAM.

See "Fitting a C^m Smooth Function to Data II" by Fefferman and Klartag, available e.g. on Fefferman's website at math.princeton.edu. Surely this notion of "computing a function" is an old idea in computer science and numerical analysis

Problem F2. Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite. How can we compute a function $F_0 \in C^{m-1,1}(\mathbb{R}^n)$ such that F_0 can be approximated arbitrarily closely in C^{m-1}-norm by functions F_ϵ as in Problem 1, with ϵ arbitrarily small?

Problem F3. Which mathematical theorems are relevant to Problems F1 and F2? (Think of the Brudnyi-Shvartsman finiteness principle.)

Problem F4. Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$. How can we tell whether there exists $F \in W^{m,p}(\mathbb{R}^n)$ such that $F = f$ on E? If F exists, how small can we take its norm?

[2] A linear operator $T : X|_S \to X$ is of depth N if for every $x \in \mathbb{R}^n \setminus S$ there exist a sequence of points $x_1, \ldots, x_N \in S$ and a sequence of numbers $\lambda_1, \ldots, \lambda_N$ such that

$$(Tf)(x) = \sum_{i=1}^{N} \lambda_i f(x_i).$$

How can we effectively compute such an F, with its Sobolev norm within a factor C of least possible, if E is finite? Here, C should depend only on m, n, p.

Pavel Shvartsman has answered the above math questions for $m = 1$; likely his ideas will also solve the computer science question in that case.

Problem F5. Let $E \subset \mathbb{R}^n$. How can we decide whether E is a subset of an imbedded (or immersed) compact, C^m-smooth surface of dimension k?

(Think of a Möbius strip in $\mathbb{R}^3$.)

Problem F6. Let $f : E \to \mathbb{R}$, with $E \subset \mathbb{R}^n$ finite. How small can we make

$$\sum_{x \in E} |F(x) - f(x)|^2$$

given that $\|F\|_{C^m(\mathbb{R}^n)}$ has order of magnitude at most M, where M is a given positive number?

(This question is due to **Andrea Bertozzi**).

Problem F7. Suppose we know that $F : \mathbb{R}^n \to \mathbb{R}$ has C^m-norm at most M. Suppose also that we are told the values $F(x_1), \cdots, F(x_N)$ of F at N given points. Our job is to pick additional points $x_{N+1}, \cdots, x_{N+N'}$, and then try to guess F as closely as possible, given the values $F(x_1), \cdots, F(x_{N+N'})$. We pick the points $x_{N+1}, \cdots, x_{N+N'}$ successively, and we are allowed to use $x_1, \cdots, x_{N+k-1}$ and $F(x_1), \cdots F(x_{N+k-1})$ in deciding which point to pick as x_{N+k}. How should we proceed?

The Problem is of course not precisely formulated. Formulate a precise version of the problem and solve it. (This question is due to **Dann Toliver**).

Problem F8. Let $\mathcal{P}$ be the space of all real polynomials of degree $\leq m$ on $\mathbb{R}^n$. Let $|\cdot|_x$, $x \in \mathbb{R}^n$, be a family of norms on $\mathcal{P}$ subject to the following two conditions:

(i) $\exists\, c_0, C_0 \in \mathbb{R}_+,\ \ \forall\, P \in \mathcal{P},\ \forall\, x \in \mathbb{R}^n$,

$$c_0 \max_{|\alpha| \leq m} |\partial^\alpha P(x)| \leq |P|_x \leq C_0 \max_{|\alpha| \leq m} |\partial^\alpha P(x)|,$$

(ii) $\exists\, C_1 \in \mathbb{R}_+,\ \forall\, x, h \in \mathbb{R}^n,\ \ |h| \leq 1, \forall\, P \in \mathcal{P}\ \ |P|_{x+h} \leq (1 + C_1|h|)|P|_x$.

Let Ω be an open subset of $\mathbb{R}^n$, let $F \in C^m(\Omega)$, define

$$\|F\|_{C^m(\Omega)} = \sup_{x \in \Omega} |J_x(F)|_x.$$

Now consider a simple case $n = m = 2$, Q is a unit square (with sides parallel to the coordinate axis) centered at the origin. Let $I = \{(x, 0) \in \mathbb{R}^n : -1.5 \leq x \leq 1.5\}$. Assume that for each $z \in I$ we are given $P^z \in \mathcal{P}$, and there exists $F_0 \in C^2(\mathbb{R}^2)$ such that

$$\forall\, z \in I\ \ J_z F_0 = P^z.$$

Assume that

$$\forall\, z \in I\ \ |P^z|_z \leq 1.$$

Given $\epsilon > 0$, does there exist $F^\epsilon \in C^2(\mathbb{R}^2)$ such that

$$\forall\, z \in I\ \ J_z F^\epsilon = P^z,$$

and

$$\|F^\epsilon\|_{C^2(Q)} \le 1+\epsilon?$$

Problems added in July of 2009.
Disclaimer: some of the problems below arose in discussions with other people.

Interpolation on Sobolev spaces

Problem F9a. Let $E \subset \mathbb{R}^2$. Characterize $W^{2,p}(\mathbb{R}^2)|_E$, where p is large but finite.

Recall: For $C^2(\mathbb{R}^2)$, the finiteness principle of Brudnyi-Shvartsman characterizes restrictions to E. One proof uses a Calderon-Zygmund decomposition. On each square of that decomposition, E lies inside an arc.

Problem F9b Is there an analogous CZ decomposition for $W^{2,p}$? If so, what kind of arcs arise? What can we say about $W^{2,p}(\mathbb{R}^2)|_E$ when E lies in an arc (e.g., the x-axis)?

Problem F9c If E is finite, then how many computer operations are needed to compute an essentially optimal interpolant?

Problem F9d. Let $E \subset \mathbb{R}^n$, and $f : E \to \mathbb{R}$. Compute the order of magnitude of the norm of f in $W^{1,p}(\mathbb{R}^n)|_E$, where p is large but finite. How many computer operations does it take?

Recall: Shvartsman proved that Whitney's extension of f is essentially best possible. So the problem is to compute the $W^{1,p}$ norm of Whitney's extension.

Interpolation in $C^m(\mathbb{R}^n)$ with Reasonable Constants.

Let $E \subset \mathbb{R}^n$ be finite, and let $f : E \to \mathbb{R}$. Bo'az and I gave an efficient algorithm to compute

$$\inf\{\|F\|_{C^m(\mathbb{R}^n)} : F|_E = f\},$$

up to a (multiplicative) constant C depending only on m, n.
Unfortunately, our proof gives an absurdly large C.

Problem F10a Compute the above inf up to a reasonable C.

To do so, it may be useful to understand the convex set

$$K = \{J_e(F) : \|F\|_{\dot{C}^m(\mathbb{R}^n)} \le 1, J_0(F) = 0\},$$

where $e = (1, 0, 0, \cdots, 0) \in \mathbb{R}^n$.

Problem F10b Compute explicit polyhedra K_{approx} for $m, n \le 3$ such that

$$K_{approx} \subset K \subset 1.1K_{approx}.$$

Problem F10c Give a coordinate-free version of the Fefferman-Klartag algorithms.

Note: In the above $\dot{C}^m(\mathbb{R}^n)$ denotes the space of locally C^m functions for which

$$\|F\|_{\dot{C}^m(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n}\left(\sum_{i_1,\cdots,i_m=1}^{m} |\partial_{i_1,\cdots,i_m}F(x)|^2\right)^{1/2}$$

is finite. So, we look only at the m-th derivatives of F. Polynomilas of degree less than m satisfy $\|P\|_{\dot{C}^m(\mathbb{R}^n)} = 0$. Also, $J_x(F)$ denotes the $(m-1)$ jet of F at x.

Interpolation with close-to-optimal C^m norm.

Pick your favorite norm on $C^m(\mathbb{R}^n)$.
Given $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite, let

$$\|f\| = \inf\{\|F\|_{C^m(\mathbb{R}^n)} : F|_E = f\}.$$

Simple examples show that the infimum need not be a minimum.

Problem F11a Given $\epsilon > 0$, compute $\|f\|$ up to a factor $(1+\epsilon)$, and compute an $F \in C^m(\mathbb{R}^n)$ such that $F|_E = f$, and

$$\|F\|_{C^m(\mathbb{R}^n)} \le (1+\epsilon)\|f\|.$$

How many computer operations does it take?

For the special case of $C^2(\mathbb{R}^2)$ (with appropriate C^2 norm) these tasks can be performed in $C(\epsilon)N\log N$ operations where N is the number of points in E, and $C(\epsilon)$ depends very badly on ϵ.

Problem F11b Prove an analogous result for vector valued functions in $C^2(\mathbb{R}^2)$, e.g., functions taking values in $\mathbb{R}^2$.

Problem F11c Achieve a practical computation of

$$|P|_{x_0} = \inf\{\|F\|_{C^m(\mathbb{R}^n)} : J_{x_0}(F) = P.\},$$

up to a factor $(1+\epsilon)$ for a given $\epsilon > 0$.
Here $J_{x_0}(F)$ denotes the m-th order Taylor polynomial of F at x_0.

Whitney fields

For the next problems, we use the following notation:
$\mathcal{P}$ = vector space of m-th degree polynomials in $\mathbb{R}^n$,
$J_x(F)$ = m-th order Taylor polynomial of F at x,
For $E \subset \mathbb{R}^n$ finite,

$$Wh(E) = \text{ vector space of "Whitney fields" on } E = \{\vec{P} = (P^x)_{x\in E} : \forall\, x \in E,\ P^x \in \mathcal{P}\}.$$

$$J_E(F) = (J_x(F))_{x\in E} \in Wh(E).$$

Problem F12a Given $\vec{P} \in Wh(E)$, it is known how to compute

$$\inf\{\|F\|_{C^m(\mathbb{R}^n)} : J_E = \vec{P}\},$$

up to a factor $(1+\epsilon)$, for suitable norms on $C^m(\mathbb{R}^n)$.

If E has N points, the computation takes $\exp(C/\epsilon)\cdot N\log N$ operations. Improve this to $C\epsilon^{-p}N\log N$, where p depends only on m and n.

Problem F13a We work in $C^2(\mathbb{R}^3)$ with a suitable norm. Let ϵ be small, and N large (think of ϵ fixed, N arbitrary). Let E be a lattice of N points in the unit square in the xy-plane in $\mathbb{R}^3$. Let S be a lattice of $\sim \epsilon^{-3}$ points in the unit cube in $\mathbb{R}^3$.

Given $f: E \to \mathbb{R}$ and given a real number $M>0$, let

$$\Gamma_f(S,M) = \{J_S(F) : \|F\|_{C^2(\mathbb{R}^3)} \leq M,\ F|_E = f\}.$$

Compute a convex polyhedron $\tilde{\Gamma}(S,M)$ such that

$$\Gamma_f(S,M) \subset \tilde{\Gamma}(S.M) \subset \Gamma_f(S,(1+\epsilon)M).$$

Can it be done in $C(\epsilon)N\log N$ operations? (It can be done in $C(\epsilon)N^5(\log N)^2$ operations.)

"Moves".

The motivation of the next problem is as follows:

Suppose $f: E\to\mathbb{R}$ with $E\subset\mathbb{R}^n$ finite. Let N be the number of points of E, and let $\epsilon>0$ be given. Suppose that there exists no $F\in C^m(\mathbb{R}^n)$ with norm $\leq 1+\epsilon$ such that $F|_E=f$. God knows this but we don't. The point of the next problem is that (I think) God can prove to us that there exists no $F\in C^m(\mathbb{R}^n)$ with norm ≤ 1 such that $F|_E = f$. God's proof takes at most $C(\epsilon)N$ steps, this is probably the best possible. God's proof will consist of a sequence of at most $C(\epsilon)N$ 'moves".

Let $f: E\to\mathbb{R}$ with $E\subset\mathbb{R}^n$ and let $\epsilon>0$ be given.

We present several "moves" by which we may compute convex polyhedron $K_S \subset WH(S)$ for finite $S\subset\mathbb{R}^n$, such that the following holds:

Any $F\in C^m(\mathbb{R}^n)$ with norm ≤ 1, and satisfying $F|_E = f$, must also satisfy $J_S(F)\in K_S$.

The moves are as follows:

1. We may take $S=\{x\}$ with $x\in E$, and define

$$K_S = \{J_S(F) : F(x) = f(x)\}.$$

2. Given S, it's known how to compute an "approximate unit ball" $K_{AUB}(S)\subset Wh(S)$, with the following properties:

(a) $F\in C^m(\mathbb{R}^n), \|F\|_{C^m(\mathbb{R}^n)}\leq 1$ imply $J_S(F)\in K_{AUB}(S)$,

(b) $\vec{P}\in K_{AUB}(S)$ implies that there exists $F\in C^m(\mathbb{R}^n)$ such that $\|F\|_{C^m(\mathbb{R}^n)}\leq 1+\epsilon$ and $J_S(F)=\vec{P}$,

(c) $K_{AUB}(S)$ is a convex polyhedron defined by at most $C(\epsilon)$ linear constraints.

For any S, an **"opening move"** is to take $K_S = K_{AUB}(S)$.

The **"subsequent moves"** are as follows:

3. Suppose we have already produced convex polyhedra $K_{S_1}, K_{S_2}, \cdots, K_{S_L}$ corresponding to finite subsets $S_1, S_2, \cdots, S_L\subset\mathbb{R}^n$, respectively.

Then we may take any $S \subset S_1 \cup S_2 \cup \cdots \cup S_L$, and define

$$K_S = \{\vec{P}|_S : \vec{P} \in Wh(S_1 \cup S_2 \cup \cdots \cup S_L), \vec{P}|_{S_l} \in K_l,\ l = 1, \cdots, L\}.$$

4. If we have already produced $K_S \subset Wh(S)$, and if $K_S \subset K'_S$ for another convex polyhedron $K'_S \subset Wh(S)$, then we may pass from K_S to K'_S.

This completes the definition of "Moves".

Problem F14 – CONJECTURE: Given f, E, ϵ as above, with E containing N points, there exists a sequence of at most $C(\epsilon)N$ moves, with the following properties:

(a) All the sets S arising in our moves have at most $C(\epsilon)$ points,

(b) All the convex polyhedra K_S arising in our moves are defined by at most $C(\epsilon)$ constraints,

(c) Given $\vec{P} \in K_{\overline{S}}$, there exists $F \in C^m(\mathbb{R}^n)$, with norm at most $1+\epsilon$, such that $J_{\overline{S}}(F) = \vec{P}$. Here $\overline{S}$ is the set arising in out last "move".

In particular, if (as God knows) there exists no $F \in C^m(\mathbb{R}^n)$ of norm at most $1+\epsilon$ such that $F|_E = f$, then the last move will produce the convex set $K_{\overline{S}} = \emptyset \subset Wh(\overline{S})$.

This will prove to us that there can be no $F \in C^m(\mathbb{R}^n)$ of norm at most 1 such that $F|_E = f$.

Interpolation in $C^{m-1,1}(\mathbb{R}^n)$ with Optimal Norm.

Given $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite, and given your favorite norm on $C^{m-1,1}(\mathbb{R}^n)$, there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ such that $F|_E = f$ with $\|F\|_{C^{m-1,1}(\mathbb{R}^n)}$ as small as possible.

Problem F15 Compute the norm of such an F.

What does it mean to compute F itself?

Give an efficient algorithm to compute such an F.

Other Problems on Interpolation in $C^m(\mathbb{R}^n)$.

Problem F16a Can we interpolate in $C^m(\mathbb{R}^n)$ with C^m-norm of the least possible order of magnitude, using SPLINES, i.e., $\sum_\nu P_\nu \mathbb{I}_{Q_\nu}$, where P_ν is a polynomial of degree D, the Q_ν form a partition of some big cube into subcubes, and D depends only on m, n? (The algorithm of Bo'az and Charlie gives an interpolant of the form

$$\sum_\nu \frac{P_\nu}{S_\nu} \mathbb{I}_{Q_\nu},$$

where P_ν, S_ν are polynomials of degree D, and $c < S_\nu < C$ on Q_ν.)

Fix a modulus of continuity ω. For $E \subset \mathbb{R}^n$, write $C^{m,\omega}(E)$ for the space of restrictions $F|_E$ of functions $F \in C^{m,\omega}(\mathbb{R}^n)$. Define the norm on $C^{m,\omega}(E)$

$$\|f\|_{C^{m,\omega}(E)} = \inf\{\|F\|_{C^{m,\omega}(\mathbb{R}^n)} : F|_E = f\}.$$

If E is finite but arbitrarily large then there exists an interpolation operator $T : C^{m,\omega}(E) \to C^{m,\omega}(\mathbb{R}^n)$ of "bounded depth". That is

(a) $\|Tf\|_{C^{m,\omega}(\mathbb{R}^n)} \le C\|f\|_{C^{m,\omega}(E)}$,

(b) For each $x \in \mathbb{R}^n$ there exist $y_1, \cdots, y_k \in E$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$ such that

$$\forall\, f \in C^{m,\omega}(E) \quad (Tf)(x) = \sum_{i=1}^{k} \lambda_i f(y_i),$$

and $k \leq D \equiv$ "depth", a large constant depending on n, m,

(c) $Tf|_E = f$ for each $f \in C^{m,\omega}(E)$.

Suppose E is closed but infinite. It's known that there exists T satisfying (a) and (c), but not (b).

Problem 16b. Can we find T satisfying (a), (b) and (c)?

With C^m in place of $C^{m,\omega}$, the answer is NO, but I guess the answer for $C^{m,\omega}$ is YES.

Problems Suggesting Links to Algebraic Geometry

Problem F17a If $E \subset \mathbb{R}^n$, let $I_x(E) = \{J_x(F) : F \in C^m(\mathbb{R}^n), F|_E = 0\}$. Then $I_x(E)$ is an ideal in the ring of jets at x. What ideals arise as $I_x(E)$? Can we say anything non-trivial about $I_x(E)$?

Let $E \subset \mathbb{R}^n$. For $x \in \mathbb{R}^n$, let

$$K_x(E) = \{J_x(F) : F \in C^m(\mathbb{R}^n), \|F\|_{C^m(\mathbb{R}^n)} \leq 1, F|_E = 0\}.$$

Then $K_x(E)$ is Whitney convex, with Whitney constant $\leq C$.

Problem F17b Which Whitney convex sets (with Whitney constant $\leq C$) arise as $K_x(E)$?

Here we are not interested in the exact size and shape of $K_x(E)$. Rather, we view two convex symmetric sets K, K', containing 0, as essentially equivalent if $cK \subset K' \subset CK$ with c, C depending only on n, m.

Can we say anything about $K_x(E)$ other than that it is Whitney convex?

Problem F18 Clarify the connections (if there are any) between the Brudnyi-Shvartsman finiteness principle and blowing up singularities.

Fitting a Submanifold to Data.

Fix $m, n, k \geq 1$. Let $E \subset \mathbb{R}^n$ (maybe we require E finite).

Problem F19 How can we tell whether there exists a C^m-smooth submanifold $M \subset \mathbb{R}^n$ of dimension k (not necessarily a graph, even when $k = n-1$) such that $E \subset M$?

What's the right formulation of the problem?

Are there sets analogous to $\Gamma(x, M)$ that play a role in the answer?

If so, what properties of those sets play the role of convexity of $\Gamma(x, M)$?

5. Pavel Shvartsman's problems

1. The Finiteness Property. We let $C^{k,\omega}(\mathbb{R}^n)$ denote the space of all function $f \in C^k(\mathbb{R}^n)$ whose partial derivatives of order k satisfy the Lipschitz condition (with respect to the metric $\omega(\|x-y\|)$). Recall that this space possesses the following "finiteness property":

There is a positive integer $N = N(k,n)$ such that the following is true: Let f be a function defined on a closed subset $S \subset \mathbb{R}^n$. Suppose that the restriction $f|_{S'}$ of f to an arbitrary subset $S' \subset S$ consisting of at most N points can be extended to a function $F_{S'} \in C^{k,\omega}(\mathbb{R}^n)$ with norm $\|F_{S'}\|_{C^{k,\omega}(\mathbb{R}^n)} \le 1$.

Then the function f itself can be extended to a function $F \in C^{k,\omega}(\mathbb{R}^n)$ with $\|F\|_{C^{k,\omega}(\mathbb{R}^n)} \le \gamma$ where $\gamma = \gamma(n,k)$ is a constant depending only on n and k.

We call the number N appearing in formulations of finiteness properties *"the finiteness number"*.

H. Whitney [34] characterized the restriction of the space $C^k(\mathbb{R})$, $k \ge 1$, to an arbitrary subset $S \subset \mathbb{R}$ in terms of divided differences of functions. An application of Whitney's method to the space $C^{k,\omega}(\mathbb{R})$ implies the finiteness property for this space with the finiteness number $N(k,1) = k+2$.

Brudnyi and Shvartsman [27, 6] proved that the sharp value of the finiteness number for the space $C^{1,\omega}(\mathbb{R}^n)$ is $N(1,n) = 3 \cdot 2^{n-1}$. Fefferman [11, 13] showed that the finiteness property holds for every $k, n \ge 1$. An upper bound for the finiteness number $N(k,n)$ given in [11, 13] is

$$N(k,n) \le (\dim \mathcal{P}_k + 1)^{3 \cdot 2^{\dim \mathcal{P}_k}}.$$

Here $\mathcal{P}_k$ stands for the space of polynomials of degree at most k defined on $\mathbb{R}^n$. (Recall that $\dim \mathcal{P}_k = \binom{n+k}{k}$.)

Basing on this estimate of $N(k,n)$ Bierstone and Milman [2] and Shvartsman [31] proved that the finiteness property for $C^{k,\omega}(\mathbb{R}^n)$ holds with the finiteness number $N(k,n) = 2^{\dim \mathcal{P}_k}$.

Problem S1. *Find the sharp value of the "finiteness number" $N = N(k,n)$ for the space $C^{k,\omega}(\mathbb{R}^n)$ for $k > 1$.*

In [31] we conjectured the following:

Conjecture S1.1 *The sharp value of the finiteness number for $C^{k,\omega}(\mathbb{R}^n)$ equals*

$$N(k,n) = \prod_{m=0}^{k} (k-m+2)^{\binom{n+m-2}{m}}.$$

In particular, in two dimensional case the conjecture states that $N(k,2) = (k+2)!$. Thus, the "simplest" case where the problem is open is the case of the space $C^{2,\omega}(\mathbb{R}^2)$.

Is it true that in this case $N(2,2) = 24$?

2. Trace criterion for the space $C^{k,\omega}(\mathbb{R}^n)$. As we have noted above, H. Whitney [34] characterized the restriction $C^k(\mathbb{R})|_S$ in terms of divided differences of functions. A similar intrinsic characterization of the trace space $C^{k,\omega}(\mathbb{R})|_S$ has been obtained by Merrien [25].

Recall a trace criterion for the space $C^{1,\omega}(\mathbb{R}^2)\,|_S$, $S \subset \mathbb{R}^2$, presented in author's papers [29]. (Note that by the finiteness theorem this criterion is expressed in terms of exactly 6 (arbitrary!) points of S.)

Let $Z \subset S$ be an arbitrary set consisting of three points and let θ_Z be the biggest angle of the triangle whose vertices are the points of Z. We let P_Z denote the affine polynomial interpolating f at the points of Z.

A locally bounded function f is in $C^{1,\omega}(\mathbb{R}^2)\,|_S$ if and only if there exists $\lambda > 0$ such that the following inequalities hold:

(i). for every subset $Z = \{z_0, z_1, z_2\} \subset X$ such that z_1 belongs to the line segment $[z_0, z_2]$

$$\left| \frac{f(z_0) - f(z_1)}{\|z_0 - z_1\|} - \frac{f(z_1) - f(z_2)}{\|z_1 - z_2\|} \right| \le \lambda\omega(\|z_0 - z_2\|)\ ;$$

(ii). for every pair of subsets $Z_1, Z_2 \subset S$ each consisting of three non-collinear points

$$\|\nabla P_{Z_1} - \nabla P_{Z_2}\| \ \le\ \lambda \left\{ \frac{\omega(\operatorname{diam} Z_1)}{\sin\theta_{Z_1}} + \frac{\omega(\operatorname{diam} Z_2)}{\sin\theta_{Z_2}} + \omega\left(\operatorname{diam} Z_1 \cup Z_2\right) \right\}.$$

Moreover, $\|f\|_{C^{1,\omega}(\mathbb{R}^2)|_S} \sim \inf\lambda$.

Observe that the proofs of the finiteness property for the space $C^{k,\omega}(\mathbb{R}^n)$ given in [27, 6] ($k = 1$) and [11, 13] (arbitrary $k, n \ge 1$) are constructive. However, straightforward application of these algorithms to $N(k,n)$-element sets leads to very complicated trace criterions.

Problem S2. *Find a trace criterion for the space $C^{k,\omega}(\mathbb{R}^n)$ for $n > 1$.*

We mean that this criterion uses only the values of a function on the set S and certain geometric characteristics of S. Thus, the "simplest" case where the problem is open is the case of the space $C^{1,\omega}(\mathbb{R}^3)$. Observe that, by the "finiteness property" for this space, the corresponding criterion should be expressed in terms of exactly 12 (arbitrary!) points of $S \subset \mathbb{R}^3$.

3. The Whitney extension problem for the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$. In this section we present several Whitney-type problems formulated in the joint paper with Yuri Brudnyi [3]. Let m be a non-negative integer. We let Ω_m denote the class of non-decreasing continuous functions $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0) = 0$ and the function $\omega(t)/t^m$ is non-increasing. Given non-negative integers k and m and $\omega \in \Omega_m$ we define the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$ as follows: a function $f \in C^k(\mathbb{R}^n)$ belongs to $C^k\Lambda^m_\omega(\mathbb{R}^n)$ if there exists $\lambda > 0$ such that for every multi-index α, $|\alpha| = k$, and every $x, h \in \mathbb{R}^n$ we have $|\Delta^m_h(D^\alpha f)(x)| \le \lambda\omega(\|h\|)$.

Here as usual $\Delta_h^m f$ denotes the m-th difference of a function f of step h:

$$\Delta_h^m f(x) := \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x+ih).$$

$C^k\Lambda_\omega^m(\mathbb{R}^n)$ is normed by

$$\|f\|_{C^k\Lambda_\omega^m(\mathbb{R}^n)} := \sum_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha f(x)| + \sum_{|\alpha|=k} \sup_{x,h\in\mathbb{R}^n} \frac{|\Delta_h^m(D^\alpha f)(x)|}{\omega(\|h\|)}.$$

In particular, for $m = 1$ and $\omega \in \Omega_1$ the space $C^k\Lambda_\omega^1(\mathbb{R}^n)$ coincides with the space $C^{k,\omega}(\mathbb{R}^n)$. In turn, the space $\Lambda_\omega^m(\mathbb{R}^n) := C^0\Lambda_\omega^m(\mathbb{R}^n)$, $\omega \in \Omega_m$, coincides with the generalized Zygmund space of bounded functions f on $\mathbb{R}^n$ whose modulus of smoothness of order m, $\omega_m(t; f)$, satisfies the inequality $\omega_m(t; f) \le \lambda\omega(t)$, $t \ge 0$. In particular, the space $\Lambda_\omega^2(\mathbb{R}^n)$ with $\omega(t) = t$ is the classical Zygmund space of bounded functions satisfying the Zygmund condition: there is $\lambda > 0$ such that for all $x, y \in \mathbb{R}^n$

$$|f(x) - 2f\left(\tfrac{x+y}{2}\right) + f(y)| \le \lambda\|x-y\|.$$

Consider three Whitney's type problems for the space $C^k\Lambda_\omega^m(\mathbb{R}^n)$.

Problem S3. *Whether the space $C^k\Lambda_\omega^m(\mathbb{R}^n)$ possesses the finiteness property?*

For $n = 1$ the finiteness property follows from the results of Merrien [23], Jonsson [21], Shevchuk [26] and Galan [17]. If $n > 1$, the answer is positive for $m = 1$ (see Section 1), and for $k = 0, m = 2$, i.e., for the Zygmund space, see [27] (in this case the optimal finiteness number $N = 3 \cdot 2^{n-1}$ is the same as for $C^{1,\omega}(\mathbb{R}^n)$.)

Problem S4. *Given closed subset $S \subset \mathbb{R}^n$, does there exist a linear continuous extension operator $T : C^k\Lambda_\omega^m(\mathbb{R}^n)|_S \to C^k\Lambda_\omega^m(\mathbb{R}^n)$?*

The answer is positive for $m = 1$ and arbitrary $n > 1$ (for $k = 1$ see Brudnyi and Shvartsman [5], for $k > 1$ see Fefferman [12, 15]); for $k = 0, m = 2$ see [5].

Problem S5. *Find a trace criterion for the space $C^k\Lambda_\omega^m(\mathbb{R}^n)$.*

We have such a criterion only for the space $C^{1,\omega}(\mathbb{R}^2)$ (see Section 2) and $\Lambda_\omega^2(\mathbb{R}^2)$ [29].

4. Sobolev Extension Domains. Given positive integer k and $p \ge 1$, a domain Ω in $\mathbb{R}^n$ is said to be Sobolev W_p^k-extension domain if the following isomorphism

$$W_p^k(\Omega) = W_p^k(\mathbb{R}^n)|_\Omega$$

holds. In other words, Ω is a Sobolev extension domain if every Sobolev function $f \in W_p^k(\Omega)$ can be extended to a Sobolev W_p^k-function F defined on all of $\mathbb{R}^n$.

For instance, Lipschitz domains (Calderón [9], $1 < p < \infty$, Stein [33], $p = 1, \infty$) in $\mathbb{R}^n$ are W_p^k-extension domains for every $p \in [1, \infty]$ and every $k \in N$. Jones [20] introduced a wider class of (ε, δ)-domains and proved that every (ε, δ)-domain is a Sobolev W_p^k-extension domain in $\mathbf{R}^n$ for every $k \ge 1$ and every $p \ge 1$. Burago and Maz'ya [8], [23], Ch.

6, described extension domains for the space $BV(\mathbb{R}^n)$ of functions whose distributional derivatives of the first order are finite Radon measures; see also [1].

Goldshtein, Latfullin and Vodopyanov [18, 19] proved that every simply connected bounded planar domain is a W_2^1-extension domain if and only if Ω is a quasi-disk, i.e., the image of a disk under a quasi-conformal mapping of the plane onto itself (note that quasi-disk is an (ε, δ)-domain). Maz'ja [23, 24] gave an example of a simply connected domain $\Omega \subset \mathbb{R}^2$ such that Ω is a W_p^1-extension domain for every $p \in [1, 2)$, while $\mathbb{R}^2 \setminus \bar{\Omega}$ is a W_p^1-extension domain for all $p > 2$. However Ω is not an $(\varepsilon, \delta)-$domain for any ε and δ.

The case $p = \infty$ has been studied by Whitney [35] who proved that quasi-Euclidean domains are W_∞^k-extension domains for every $k \geq 1$ (Ω is quasi-Euclidean if its inner (or geodesic) metric is equivalent to the Euclidean distance). Zobin [37] showed that every finitely connected bounded planar W_∞^k-extension domain is quasi-Euclidean. He also showed, see [36], that for every $k \geq 2$ there exists a bounded planar W_∞^k-extension domain which is not quasi-Euclidean.

Extension properties of α-subhyperbolic domains in $\mathbb{R}^n$ determined by certain inner metrics has been studied by Koskela [22] and Shvartsman [32] (observe that for $\alpha = 0$ this class coincides with (ε, δ)-domains, and for $\alpha = 1$ with quasi-Euclidean domains). In [32] it was shown that every $\frac{p-n}{p-1}$-subhyperbolic domain $\Omega \subset \mathbb{R}^n$ is a Sobolev W_q^k-extension domain provided $p > n$ and $q > p^*$ where $p^* \in (n, p)$ depends only on p, n and Ω; for $k = 1$ and $q > p$ this has been proved in [22].

Combining this result with the necessity condition for a finitely connected bounded domain $\Omega \subset \mathbb{R}^2$ due to Buckley and Koskela [7], we obtain that such a domain is a Sobolev W_p^1-extension domain for some $p > 2$ iff Ω is $\frac{p-2}{p-1}$-subhyperbolic.

Problem S6. *Given $p \geq 1, n > 1, k \geq 1$ find a geometric description of the class of Sobolev W_p^k-extension domains in $\mathbb{R}^n$.*

Thus, the "simplest" unknown case is the case of a simply connected planar W_p^1-extension domain with $p \in [1, 2)$.

5. A geometrical background of the finiteness property: some connections with convex geometry. Let us recall several problems posed in the joint paper with Yuri Brudnyi [4].

The proof of the finiteness property for the space $C^{1,\omega}(\mathbb{R}^n)$ presented in [27, 6] is based on the following geometrical result expressed in terms of *set-valued maps* and their *Lipschitz selections*.

Let $(\mathcal{M}, d)$ be a metric space and let F be a set-valued mapping from $\mathcal{M}$ into the family $Conv(\mathbb{R}^n)$ of all convex compact subsets of $\mathbb{R}^n$.

Problem S7. *Find conditions on F for which it has a Lipschitz selection.*

Recall that a mapping $f : \mathcal{M} \to \mathbb{R}^n$ is said to be a Lipschitz selection of F if $f(x) \in F(x)$ for every $x \in \mathcal{M}$ and $f \in \mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)$.

Conjecture S7.1 *(*[4]*) Let F be a set-valued mapping from a metric space $(\mathcal{M}, d)$ into $Conv(\mathbb{R}^n)$. Suppose that, for every subset $\mathcal{M}' \subset \mathcal{M}$ consisting of at most 2^n points, the* *restriction* $F|_{\mathcal{M}'}$ *of* F *to* $\mathcal{M}'$ *has a Lipschitz selection* $f_{\mathcal{M}'}$ *such that* $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',\mathbb{R}^n)} \le 1$. *Then* F *has a Lipschitz selection* f *with* $\|f\|_{\mathrm{Lip}(\mathcal{M},\mathbb{R}^n)}$ *bounded by some constant* $\gamma = \gamma(n)$ *depending only on* n.

Note that in the case of the trivial pseudometric $d \equiv 0$ the conjecture is true, even with $n+1$ instead of 2^n, since in this case it is exactly the classical theorem of Helly [10]. In [28, 29, 30] we prove he following:

(1). The conjecture is true for $\mathbb{R}^2$.

(2). The conjecture is true for every finite k-point metric space $\mathcal{M}$, *but with* $\gamma = \gamma(n,k)$.

(3). If the conjecture is true for some number $N(n)$ *in place of* 2^n *then it also holds in its original form.*

(4). The conjecture is false in general if 2^n *is replaced by some number* $N(n)$ *with* $N(n) < 2^n$.

(5). The conjecture is true for set-valued maps F *which take values in the class* $\mathcal{A}(\mathbb{R}^n)$ *of all affine subsets of* $\mathbb{R}^n$. *(In case of the Whitney extension problem for* $C^{1,\omega}(\mathbb{R}^n)$ *we only need to consider this class of convex subsets of* $\mathbb{R}^n$.*)*

Thus the "simplest" case where the conjecture is open is the case of the metric space $\mathcal{M} = \{1, 2, ..., m\}$ and $n = 3$.

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6. Yosef Yomdin's Problems

Problem Y1. Given $n, m, k \in Z_+$, and a set $E \subset U$, where U is the unit ball in $\mathbb{R}^n$, such that $\#(E) \leq k$, do there exist $D, C > 0$, depending only upon n, m, k, such that for any function $f : E \to \mathbb{R}$ for which there exists $F \in C^m(\mathbb{R}^n)$, $F|_E = f$, with $\|F\|_{C^m(\mathbb{R}^n)} \leq 1$ one can find a polynomial P of degree at most D such that $P|_E = f$ and

$$\|P\|_{C^m(U)} \leq C?$$

7. Nahum Zobin's Problems

Problem Z1. Find an analog of the Whitney Theorem for tensor fields on a manifold with a connection (e.g., a Riemannian manifold). The connection is needed since otherwise one cannot compare tangent spaces at different points and therefore one cannot define the notion of a tensor field with uniformly bounded k-th derivatives.

Problem Z2. Extension of functions and tensor fields (with uniformly bounded k-th derivatives) subject to differential equations and inequalities. Extension of differential forms with uniformly bounded differentials. In particular, extension of closed bounded differential forms with preservation of closedness and boundedness.

7.1. Geometry of open domains and extensions of functions. Let Ω be an **open** subset of $\mathbb{R}^n$. Then the notion of a smooth function is well defined, and so are the spaces $C^{m,1}(\Omega)$. We say that $\Omega \in EP(m)$, if

$$C^{m,1}(\mathbb{R}^n)|_\Omega = C^{m,1}(\Omega).$$

Whitney proved that if Ω satisfies the Whitney Condition (the geodesic metric in Ω is equivalent to the Euclidean metric, this condition is also called quasi-convexity – see Gromov's book) then $\Omega \in EP(m)$ for all $m \in N$. One can rather easily show that if $\Omega \in EP(0)$ then Ω is quasi-convex. This means that $\Omega \in EP(0)$ iff Ω is quasi-convex, and the condition $\Omega \in EP(0)$ implies that $\Omega \in EP(m)$ for any $m \in N$. I have shown that for a **finitely connected planar** Ω and for any $m \in N$ the condition $\Omega \in EP(m)$ is equivalent to the condition that Ω is quasi-convex. However, I have also shown that if Ω is an infinitely connected planar domain, or a domain in $\mathbb{R}^n, n \geq 3$, then the condition $\Omega \in EP(m), m \geq 1$, does not imply that $\Omega \in EP(l)$, $l \leq m$, so the higher extension properties do not imply lower extension properties. Do lower extension properties imply higher ones? Maybe, not.

Problem Z3. Construct $\Omega_m, m \geq 1$, – an infinitely connected planar domain, or a domain in $\mathbb{R}^n, n \geq 3$, – such that $\Omega \in EP(m)$ but $\Omega \notin EP(m'), \forall\, m' > m$.

Can one save the implication

$$\Omega \in EP(m),\ m > 0, \Rightarrow \Omega \text{ is quasi-convex}$$

for infinitely connected planar domains Ω by imposing topological restrictions on Ω? Maybe.

Problem Z4. Show that if the pairwise distances between components of $\mathbb{R}^2 \setminus \Omega$ are bounded from below, and if $\Omega \in EP(m)$ for some $m \in N$, then Ω is quasi-convex.

Is there any topological cure in higher dimensions? Unlikely:

Problem Z5. For each $m \in N$ construct a domain $\Omega_m \subset \mathbb{R}^3$, homeomorphic to an open ball, with boundary smooth at all points, except of one, and such that

$$\Omega \in \left(EP(m) \setminus \bigcup_{0 \le k < m} EP(k) \right).$$