# JORDAN DECOMPOSITION, I. A GEOMETRIC APPROACH

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Abstract. We study an infinite-dimensional analog of the Jordan decomposition. This paper is devoted to the motivations and main geometric constructions involved in such Jordan decompositions. For an arbitrary bounded linear operator in a Banach space satisfying condition (DCC) (see below) we construct a broader locally convex space such that

(i) the initial Banach space is densely and continuously imbedded into this space,

(ii) the operator and all possible rational functions of the operator are continuously extendable to this space, and

(iii) all possible root vectors of the operator belong to this space.

The forthcoming part II of the paper is devoted to problems of completeness of systems of generalized root vectors.

## Introduction

The Jordan theorem on the normal form of a matrix is certainly one of the central results in linear algebra. This theorem states that root vectors of a linear operator $A$ in a finite-dimensional complex linear space $V$ form a complete system. But the natural question arises:

**Question.** *What is a Jordan decomposition in the infinite-dimensional situation?*

This problem has been very much studied and discussed [2, 3, 6–10], in particular, because of its obvious importance in analysis. In his remarkable plenary address [7] to the International Congress of Mathematicians in Moscow (1966), M. G. Krein called this problem "the Blue Bird of Functional Analysis".

¿From the very beginning, there are several difficulties that are mainly related to adequate generalizations of the main notions involved.

The following objects play the principal role in the finite-dimensional situation:

(i) $\operatorname{Spec} A$, the set of all $\lambda \in \mathbb{C}$ such that the equation $(A-\lambda\mathbb{I})x = 0$ has nontrivial solutions;

(ii) Eigenvectors = nontrivial solutions of the above equation;

(iii) $\operatorname{Spec}^k A$, the set of all $\lambda \in \mathbb{C}$ such that the equation $(A-\lambda\mathbb{I})^{k+1}x = 0$ has nontrivial solutions, i.e., such that $(A-\lambda\mathbb{I})^k x \neq 0$ (in particular, $\operatorname{Spec}^0 A = \operatorname{Spec} A$);

(iv) Root vectors = nontrivial solutions of the above equations.

The notion of spectrum can be adequately generalized to the infinite-dimensional setting, but in the initial space, the related eigenvectors may not exist. There is no general understanding what part of Spec $A$ should be attributed to $\mathrm{Spec}^k A,\ k \geq 1$, and, anyway, the related root vectors may also be missing in the initial space. In this paper we propose natural definitions for all the above notions.

In fact, the Jordan theorem has two layers. The first one is a description of the algebra generated by the operator $A$, and the second one deals with the multiplicity phenomenon of the spectrum. Here we restrict ourselves only to the first layer and study the structure of the algebra generated by $A$, leaving aside difficult problems related to the multiple spectrum (we hope to discuss them in subsequent publications). Our constructions can be partially generalized to the case of multiple spectra, but they become much more complicated.

This paper is the first part of our exposition. It is chiefly devoted to motivations and to the main geometric construction.

Our approach is closely related to the theory of generalized eigenvectors of self-adjoint operators in a Hilbert space, initiated by I.M. Gelfand and A.G. Kostuchenko [4], see also [1]. The idea of this approach is to extend, in a natural way, the action of the operator in question to a broader space and try to find the missing eigenvectors there. The following well-known example gives the flavor of the approach: consider the operator $id/dx$ in the space $L_2(\mathbb{R})$. Its spectrum is the entire real line $\mathbb{R}$, the functions $e^{i\lambda x}$ are eigenvectors of this operator, but they all belong not to the initial space $L_2(\mathbb{R})$, but to a broader space, say, the space of bounded functions. Nevertheless, we know that Fourier analysis provides a very nice and rich theory of expansion of functions in integrals over these eigenvectors.

Similar ideas (involving the study of generalized root vectors) were proposed and developed by V.P. Maslov and his school see [8, Chapter 1, Section 8]. In fact, the core of this approach is very understandable: all kinds of spectral decompositions may be reformulated and studied in terms of suitable functional calculi. This idea was deeply investigated and developed by I. Colojoara, B. Sz.-Nagy, C. Foias, F.H. Vasilescu and others (see [2, 9, 10] and references therein).

We suggest a complete implementation of this program. Namely, we construct an extension of the initial space and show that one can find all possible root vectors in the constructed space. We show that it is impossible to stay within the setting of normed spaces if one wishes to find all root vectors, and the natural home for all root vectors is a nonnormable locally convex space.

A crucial difference between the finite- and the infinite-dimensional situations is that there are no natural ways to extend an operator in a finite-dimensional space to a broader space, whereas every infinite-dimensional operator acts simultaneously in many spaces. It is very difficult to say in advance which space is more natural for this operator, and thus we do not restrict ourselves to the initial space when trying to find natural objects, in particular, eigenvectors and root vectors.

To get a better feeling of our approach, let us discuss the following example: consider the operator

$$(Af)(t) = \int_0^t f(s)ds$$

in the space of continuous functions on the interval $[0,1]$. It is well known that this operator is quasi-nilpotent, i.e., its spectrum consists of a single point 0. What are natural eigenvectors and root vectors in this situation? One can readily show that the $\delta$-functional, supported at 1, is an eigenvector of $A$. But it already belongs to a broader space, say, to the space of Borel measures on $[0,1]$. We can readily show as well that the derivatives of this $\delta$-functional are root vectors of the operator $A$. But they belong to even broader spaces, and the space of distributions (which is a nonnormable locally convex space) contains them all. One can write the Jordan decomposition for this operator, and it coincides with the expansion in Taylor series. Note that the Taylor expansion is meaningful not for all functions from the initial space of continuous functions but only for its dense lineal of analytic functions. A similar situation is well known in ordinary Fourier analysis: the Fourier decomposition holds for absolutely integrable functions only, but one has to do some regularization for other functions from $L_2(\mathbb{R})$.

In the second part of our research, we give an analytic approach to our constructions, permitting a much more precise study of the completeness problems for the generalized root vectors obtained.

After we have constructed the space where all possible root vectors of the operator $A$ live, we can pose the question whether these root vectors form a complete system. We show that the answer to this question is equivalent to the "generalized semisimplicity" of a specially constructed algebra, and we call this property "semi-intricacy". For the case in which the root vectors form a complete system (this means that the related algebra is semi-intricate), we obtain the exact analog of the Jordan decomposition formula, involving integration with respect to a "generalized measure".

The paper is organized as follows. Section 1 contains the necessary preliminaries. In Sections 2 and 3 we describe the first step in the main construction and present a theory of expansion of vectors into "integrals" over generalized eigenvectors with respect to a generalized measure. All results of these sections are actually well known, we only present them in a way suitable for further considerations. Section 4 is devoted to a geometric description of the space containing all possible root vectors.

In the second part of the paper we continue our investigations. In Section 5 we study the above construction in analytic terms and introduce some notions needed for the generalized Gelfand transform. Section 6 is devoted to the generalized Gelfand transform and to completeness problems for the generalized root vectors. In Section 7 we apply the previous considerations to obtain a Jordan decomposition for the general operator.

Preliminary versions of these results were presented at various conferences since 1983: Chernogolovka (1983), Voronezh (1985, 1991), Halle (1988), Novgorod (1989), Oberwolfach (1990), Sapporo (1990), Jerusalem (1991), and Beer-Sheva (1992). A draft of this paper [11] has circulated since 1993 as a preprint of the Max-Planck-Institute (Bonn).

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## 1. Main Notions

Let $V$, $V'$ be a pair of complex Banach spaces. For simplicity, we assume that either $V'$ is the Banach dual space for $V$ or $V$ is the Banach dual space for $V'$, and we prefer not to specify which of them is dual to which (in fact, the only thing we really require is the continuity of inversion in the algebra of bounded operators). Let $p(\cdot)$ and $p'(\cdot)$ denote the norms in the spaces $V$ and $V'$, respectively.

Consider a pair of bounded linear operators $A : V \to V$ and $A' : V' \to V'$ satisfying the usual identity: $\langle Ax, x'\rangle = \langle x, A'x'\rangle \ \forall x \in V, \forall x' \in V'$. The norms of the operators are defined as usual and coincide:

$$\begin{aligned}\|A: V \to V\| &= \sup\{|\langle Ax, x'\rangle| : p(x) \le 1,\, p'(x') \le 1\}\\ &= \sup\{|\langle x, A'x'\rangle| : p(x) \le 1,\, p'(x') \le 1\} = \|A' : V' \to V'\|.\end{aligned}$$

The spectra of operators are also defined as usual and they also coincide:

$$\begin{aligned}\operatorname{Spec} A &= \{\lambda \in \mathbb{C} : (A - \lambda\mathbb{I}) \text{ has no bounded inverse }\}\\ &= \{\lambda \in \mathbb{C} : (A' - \lambda\mathbb{I}) \text{ has no bounded inverse }\} = \operatorname{Spec} A'.\end{aligned}$$

It is well known that $\operatorname{Spec} A$ is a compact subset of $\mathbb{C}$.

**Definition 1.1.** Let $\operatorname{Rat}(A)$ be the set of all rational functions with poles contained in the set $\mathbb{C} \setminus \operatorname{Spec} A$.

**Definition 1.2.** $R(A) = \{f(A) : f(\cdot) \in \operatorname{Rat}(A)\}$.

We need an equivalent description of the set $\operatorname{Spec} A$ in terms of the algebra $R(A)$. The following assertion is well known.

**Proposition 1.3.** $\lambda \in \operatorname{Spec} A$ *if and only if for any* $f \in \operatorname{Rat}(A)$ *we have* $\|f(A)\| \ge |f(\lambda)|$.

As noted above, we consider here the situation of "multiplicity-free spectra" only, i.e., we assume that the following condition holds.

**Double Cyclicity Condition (DCC).** *There exist* $\Delta \in V$ *and* $\nabla \in V'$ *such that the linear subspace* $\{B\Delta : B \in R(A)\}$ *is* $\sigma(V, V')$*-dense in* $V$*, and the linear subspace* $\{B'\nabla : B \in R(A)\}$ *is* $\sigma(V', V)$*-dense in* $V'$.

## 2. Main construction. The first step

**Rigging.** Let $J(0)$ denote the completion of $R(A)$ with respect to the operator norm $\|\cdot\|$. Then $J(0)$ is a commutative Banach algebra.

Consider the following natural linear mappings:

$$\tau_\Delta : J(0) \to V, \quad \tau_\Delta(B) = B\Delta; \qquad \tau^\nabla : J(0) \to V', \quad \tau^\nabla(B) = B'\nabla.$$

Both mappings are continuous if we equip $V$ and $V'$ with the weak topologies $\sigma(V, V')$ and $\sigma(V', V)$, respectively ($J(0)$ is always assumed to be equipped with the operator norm topology).

Consider the dual mappings $(\tau_\Delta)' : V' \to J(0)'$ and $(\tau^\nabla)' : V \to J(0)'$, where $J(0)'$ is the Banach dual space for $J(0)$.

It immediately follows from (DCC) that $\tau_\Delta$ and $\tau^\nabla$ are injections with weakly dense ranges. This implies that $(\tau_\Delta)'$ and $(\tau^\nabla)'$ are also injections with $\sigma(J(0)', J(0))$-dense ranges. (To prove the injectivity of, say, $\tau_\Delta$, one proceeds as follows: assume that $0 \ne C \in J(0)$ and $\tau_\Delta(C) = C\Delta = 0$, then the $\sigma(V, V')$-closed set $\operatorname{Ker} C$ is neither 0 nor $V$, and, for any $B \in J(0)$, we have $B \operatorname{Ker} C \subset \operatorname{Ker} C$. Since $\Delta \in \operatorname{Ker} C$, the set $\{B\Delta : B \in J(0)\}$ is not $\sigma(V, V')$-dense in $V$, and this contradicts (DCC).)

Thus, we have obtained the following injections with weakly dense ranges:

$$J(0) \xrightarrow{\tau_\Delta} V \xrightarrow{(\tau^\nabla)'} J(0)', \qquad J(0)' \xleftarrow{(\tau_\Delta)'} V' \xleftarrow{\tau^\nabla} J(0).$$

Let $V_+$ ($V^+$) denote the lineal Im $\tau_\Delta$ (Im $\tau^\nabla$), equipped with the norm transferred from $J(0)$ by the operator $\tau_\Delta$ (respectively, $\tau^\nabla$). Then the mappings $\tau_\Delta : J(0) \to V_+$ and $\tau^\nabla : J(0) \to V^+$ are isometries.

Thus, we have obtained densely imbedded Banach spaces $V_+ \subset V$ and $V' \supset V^+$. Consider the Banach dual spaces $V^- = (V_+)'$ and $V_- = (V^+)'$. The dualities between $V_+$ and $V^-$ and between $V_-$ and $V^+$ are extensions of the initial duality between $V$ and $V'$.

The mapping $(\tau^\nabla)'$ ($(\tau_\Delta)'$) naturally extends to an isometry, still denoted by $(\tau^\nabla)'$ ($(\tau_\Delta)'$), between $V_-$ and $J(0)'$ (between $V^-$ and $J(0)'$, respectively). Thus, we have the isometries $(\tau^\nabla)' : V_- \to J(0)'$ and $(\tau_\Delta)' : V^- \to J(0)'$. This gives us weakly dense inclusions $V_+ \subset V \subset V_-$ and $V^- \supset V' \supset V^+$.

**Proposition 2.1.** *For any $B, C \in J(0)$ the following statements hold:*

(i) $BV_+ \subset V_+$, $B'V^+ \subset V^+$, $\tau_\Delta(BC) = B\tau_\Delta(C)$, $\tau^\nabla(CB) = B'\tau^\nabla(C)$;

(ii) $\|B : V_+ \to V_+\| = \|B : V \to V\| = \|B' : V' \to V'\| = \|B' : V^+ \to V^+\|$;

(iii) *the action of the operator $B$ on $V$ is $\sigma(V_-, V^+)$-continuously extendable to $V_-$, and $\|B : V_- \to V_-\| = \|B : V \to V\|$; the action of the operator $B'$ on $V'$ is $\sigma(V^-, V_+)$-continuously extendable to $V^-$ and $\|B' : V^- \to V^-\| = \|B' : V' \to V'\|$.*

*Proof.* Immediate.

**Corollary 2.2.** *For any $B \in J(0)$ we have*

$$\begin{aligned}\operatorname{Spec} B &= \operatorname{Spec}\{B : V \to V\} = \operatorname{Spec}\{B : V_+ \to V_+\} = \operatorname{Spec}\{B : V_- \to V_-\} \\ &= \operatorname{Spec}\{B' : V' \to V'\} = \operatorname{Spec}\{B' : V^+ \to V^+\} = \operatorname{Spec}\{B' : V^- \to V^-\}.\end{aligned}$$

Thus, all operators from $J(0)$ have been continuously extended to a broader space $V_-$ without changing their norms, and hence without changing their spectra. This new space is more natural for these operators, and this is confirmed, in particular, by Theorem 2.5 below. This theorem is a reformulation of the following simple and well-known result [5, Section 15]).

**Proposition 2.3.** $\lambda \in \operatorname{Spec} A$ *if and only if there exists a (unique) nontrivial multiplicative functional $\varphi_\lambda \in J(0)'$ such that $\varphi_\lambda(B(A - \lambda\mathbb{I})) = 0$ $\forall B \in J(0)$.*

*Remark 2.4.* Proposition 2.3 claims that the functionals $\varphi_\lambda \in J(0)'$ are eigenvectors for the coregular action of $J(0)$ on $J(0)'$ (the coregular action is the action dual to the regular action of $J(0)$ on itself).

**Theorem 2.5.** $\lambda \in \operatorname{Spec} A$ *if and only if there exists $e_\lambda \in V_-$ such that $e_\lambda \neq 0$ and $Ae_\lambda = \lambda e_\lambda$.*

*Proof.* Let $\varphi_\lambda \in J(0)'$ be the nontrivial multiplicative functional described in Proposition 2.3. Take $e_\lambda = {(\tau^\nabla)'}^{-1}(\varphi_\lambda)$, then $e_\lambda \in V_-$, and we have $e_\lambda \neq 0$

since $(\tau^\nabla)'$ is an isometry. Take any $x' \in V^+$, then $x' = \tau^\nabla(B)$ for some $B \in J(0)$. We obtain

$$\begin{aligned}\langle (A-\lambda\mathbb{I})e_\lambda, x'\rangle &= \langle (A-\lambda\mathbb{I})(\tau^\nabla)'^{-1}(\varphi_\lambda), \tau^\nabla(B)\rangle = \langle (\tau^\nabla)'^{-1}(\varphi_\lambda), (A'-\lambda\mathbb{I})\tau^\nabla(B)\rangle \\ &= \langle (\tau^\nabla)'^{-1}(\varphi_\lambda), \tau^\nabla(B(A-\lambda\mathbb{I}))\rangle = \varphi_\lambda(B(A-\lambda\mathbb{I})) = 0,\end{aligned}$$

and thus $(A-\lambda\mathbb{I})e_\lambda = 0$.

*Remark 2.6.* Note that $\langle e_\lambda, \nabla\rangle = \langle (\tau^\nabla)'^{-1}\varphi_\lambda, (\tau^\nabla\mathbb{I})\rangle = \varphi_\lambda(\mathbb{I}) = 1$, and therefore we may always assume that $e_\lambda$ is normalized by the condition $\langle e_\lambda, \nabla\rangle = 1$ $\forall\lambda \in \operatorname{Spec} A$.

*Remark 2.7.* If we wish to construct only the space $V_-$ (without $V^-$) and find the eigenvectors $e_\lambda$ in $V_-$, then we do not need a vector $\Delta \in V$. This construction needs only the $R(A')$-cyclic vector $\nabla \in V'$. This already enables us to construct the injections $\tau^\nabla$ and $(\tau^\nabla)'$ with weakly dense ranges, and obtain the eigenvectors $e_\lambda$. This remark will be used in Section 4.

## 3. Eigenvector expansions

**Completeness of the System of Generalized Eigenvectors.** We obtained generalized eigenvectors $\{e_\lambda, \lambda \in \operatorname{Spec} A\}$ belonging to $V_-$, and now we can pose and study the problem of completeness for this system of vectors. We consider the completeness in the weakest possible topology, namely, in the weak topology $\sigma(V_-, V^+)$. This is naturally equivalent to the problem of the $\sigma(J(0)', J(0))$-completeness for the system of elements $\{\varphi_\lambda : \lambda \in \operatorname{Spec} A\}$ in $J(0)'$.

Let $C_H(\operatorname{Spec} A)$ denote the Banach algebra of functions that are continuous on $\operatorname{Spec} A$ and holomorphic in its interior. Consider the mapping

$$R(A) \ni f(A) \mapsto f|_{\operatorname{Spec} A} \in C_H(\operatorname{Spec} A).$$

The inequality of Proposition 1.6, $\sup_{\mu\in\operatorname{Spec} A} |f(\mu)| \le \|f(A)\|$, shows that the mapping is well defined and is continuously extendable to the so-called *Gelfand homomorphism* $\wedge : J(0) \to C_H(\operatorname{Spec} A)$.

*Remark 3.1.* It is well known that in this situation the spectrum of the operator $A$ can be identified with the space $\mathfrak{M}(J(0))$ of maximal ideals (= the space of nontrivial characters) of the algebra $J(0)$, and the function $\widehat{A} : \mathfrak{M}(J(0)) \to \operatorname{Spec} A$ provides the necessary identification. A version of this identification was already used in Proposition 2.3: namely, the mapping $\lambda \to \varphi_\lambda$ is a one-to-one correspondence between $\operatorname{Spec} A$ and the set of nontrivial characters on $J(0)$.

The Gelfand homomorphism can be also described as follows: $\widehat{B}(\lambda) = \varphi_\lambda(B)$ for every $B \in J(0)$ and every $\lambda \in \operatorname{Spec} A$.

The subspace $\operatorname{Ker}\wedge$ is called the *radical* of the Banach algebra. A commutative Banach algebra is said to be *semisimple* if its Gelfand homomorphism is injective, i.e., the radical $\operatorname{Ker}\wedge$ is trivial.

**Theorem 3.2.** *The system $\{\varphi_\lambda : \lambda \in \operatorname{Spec} A\}$ is $\sigma(J(0)', J(0))$-complete in $J(0)'$ if and only if the algebra $J(0)$ is semisimple.*

*Proof.* The system $\{\varphi_\lambda : \lambda \in \operatorname{Spec} A\}$ is $\sigma(J(0)', J(0))$-complete if and only if there is no $B \in J(0)$ such that $B \neq 0$ and $\varphi_\lambda(B) = 0$ for every $\lambda \in \operatorname{Spec} A$. This is equivalent to the condition that $\operatorname{Ker}\wedge = \{0\}$.

**A Refinement of the Completeness Condition.** In fact, we can drop all eigenvectors arising from the interior points of the spectrum.

Let $\partial \operatorname{Spec} A$ be the boundary of $\operatorname{Spec} A$. Then the restriction mapping $r : C_H(\operatorname{Spec} A) \to C(\partial \operatorname{Spec} A)$ is an isometric imbedding. Therefore, the mappings $\wedge$ and $r \circ \wedge$ are either both injective or both noninjective. This immediately leads to the following result.

**Proposition 3.3.** *The systems of generalized eigenvectors* $\{\varphi_\lambda : \lambda \in \operatorname{Spec} A\}$ *and* $\{\varphi_\lambda : \lambda \in \partial \operatorname{Spec} A\}$ *are either both* $\sigma(J(0)', J(0))$*-complete or both* $\sigma(J(0)', J(0))$*-incomplete.*

*Remark 3.4.* The mapping $r \circ \wedge : J(0) \to C(\partial \operatorname{Spec} A)$ is also called the *Gelfand transform* and if this causes no confusion, it is denoted by the same symbol $\wedge$.

**Generalized Eigenvector Decomposition.** Assume that the system of generalized eigenvectors $\{\varphi_\lambda : \lambda \in \partial \operatorname{Spec} A\}$ is $\sigma(J(0)', J(0))$-complete. By Theorem 3.2 and Proposition 3.3, this holds if and only if the algebra $J(0)$ is semisimple, i.e., if and only if the Gelfand homomorphism $\wedge : J(0) \to C(\partial \operatorname{Spec} A)$ is injective.

Consider the dual mapping $\wedge' : \operatorname{Meas}(\partial \operatorname{Spec} A) \to J(0)'$, where $\operatorname{Meas}(\partial \operatorname{Spec} A)$ is the space of Borel measures on $\partial \operatorname{Spec} A$, which is the Banach dual space for $C(\partial \operatorname{Spec} A)$. Since $\wedge$ is injective, it follows that the range of $\wedge'$ is $\sigma(J(0)', J(0))$-dense. Therefore, each element of $J(0)'$ can be approximated by a net of elements of the type $\wedge'(\mu_\alpha)$, where $\{\mu_\alpha\}$ is a net of measures. Therefore, we can treat the elements of $J(0)'$ as the $\wedge'$-images of "generalized measures" on $\partial \operatorname{Spec} A$ (as functionals on $\operatorname{Im} \wedge$, which is equipped with the norm transferred from $J(0)$ by the mapping $\wedge$).

**Theorem 3.5.** *Assume that the set of eigenvectors* $\{\varphi_\lambda : \lambda \in \partial \operatorname{Spec} A\}$ *is* $\sigma(J(0)', J(0))$*-complete (in other words, the algebra* $J(0)$ *is semisimple). Then for every* $\Phi \in J(0)'$*, there exists a generalized measure* $d\mu_\Phi$ *on* $\partial \operatorname{Spec} A$ *(= a bounded functional on the subalgebra* $\operatorname{Im} \wedge$ *= the limit of a net of measures* $d\mu_{\alpha,\Phi}$ *on* $\partial \operatorname{Spec} A$*) such that for every* $B \in J(0)$ *we have*

$$\Phi(B) = \int_{\partial \operatorname{Spec} A} \varphi_\lambda(B) d\mu_\Phi(\lambda) \ \left(= \lim_\alpha \int_{\partial \operatorname{Spec} A} \varphi_\lambda(B) d\mu_{\alpha,\Phi}(\lambda)\right).$$

*Proof.* The mapping $\wedge$ is an isomorphism between the algebras $J(0)$ and $\operatorname{Im} \wedge$. The generalized measure $d\mu_\Phi$ in question is just $(\wedge')^{-1}(\Phi)$, as can readily be verified:

$$\Phi(B) = (\wedge')^{-1}(\Phi)(\widehat{B}) = \int_{\partial \operatorname{Spec} A} \widehat{B}(\lambda) d\mu_\Phi(\lambda) = \int_{\partial \operatorname{Spec} A} \varphi_\lambda(B) d\mu_\Phi(\lambda).$$

*Remark 3.6.* The above formula may be treated as an eigenvector decomposition, where the integral is understood in the $\sigma(J(0)', J(0))$-sense:

$$\Phi = \int_{\partial \operatorname{Spec} A} \varphi_\lambda \, d\mu_\Phi(\lambda).$$

Now we return to the spaces $V$, $V_\pm$, and $V^\pm$. As above, $e_\lambda$ ($e^\lambda$) denote the generalized eigenvectors of $A$ (of $A'$) ($\lambda \in \operatorname{Spec} A = \operatorname{Spec} A'$; $e_\lambda \in V_-$, $e^\lambda \in V^-$). The eigenvectors are normalized as usual: $\langle e_\lambda, \nabla \rangle = 1$, $\langle \Delta, e^\lambda \rangle = 1 \ \forall \lambda \in \operatorname{Spec} A$.

We want to decompose an element of $V$ into an "integrals" of $e_\lambda$ over $\partial \operatorname{Spec} A$. Generally speaking, this is impossible for an arbitrary element of $V$, but it turns out to be possible for elements of the dense lineal $V_+$.

**Theorem 3.7.** *There exists a generalized measure $d\mu_{\Delta,\nabla}$ on $\partial\,\mathrm{Spec}\,A$ (= a bounded functional on the subalgebra $\mathrm{Im}\wedge\subset C(\partial\,\mathrm{Spec}\,A)$) such that, for every $x\in V_+$ and every $x'\in V^+$, we have*

$$\langle x,x'\rangle = \int_{\partial\,\mathrm{Spec}\,A}\langle x,e^\lambda\rangle\langle e_\lambda,x'\rangle\;d\mu_{\Delta,\nabla}(\lambda)\qquad\left(or,\;\; x=\int_{\partial\,\mathrm{Spec}\,A}\langle x,e^\lambda\rangle\,e_\lambda\;d\mu_{\Delta,\nabla}(\lambda)\right).$$

*Proof.* Since $x\in V_+$ and $x'\in V^+$, it follows that there exist $B,C\in J(0)$ such that $x=B\Delta$, $x'=C'\nabla$. Consider the functional on $J(0)$ given by $D\mapsto\langle D\Delta,\nabla\rangle$. This linear functional is clearly bounded, and therefore there exists a generalized measure $d\mu_{\Delta,\nabla}$ on $\partial\,\mathrm{Spec}\,A$ such that

$$\langle D\nabla,\Delta\rangle = \int_{\partial\,\mathrm{Spec}\,A}\widehat{D}(\lambda)d\mu_{\Delta,\nabla}(\lambda).$$

By the normalization conditions we have

$$\begin{aligned}\langle x,x'\rangle &= \langle B\Delta,C'\nabla\rangle = \langle CB\Delta,\nabla\rangle = \int\limits_{\partial\,\mathrm{Spec}\,A}(\widehat{CB})(\lambda)\;d\mu_{\Delta,\nabla}(\lambda) = \int\limits_{\partial\,\mathrm{Spec}\,A}\widehat{C}(\lambda)\widehat{B}(\lambda)\;d\mu_{\Delta,\nabla}(\lambda)\\ &= \int\limits_{\partial\,\mathrm{Spec}\,A}\widehat{C}(\lambda)\langle e_\lambda,\nabla\rangle\widehat{B}(\lambda)\langle\Delta,e^\lambda\rangle\;d\mu_{\Delta,\nabla}(\lambda) = \int\limits_{\partial\,\mathrm{Spec}\,A}\langle Ce_\lambda,\nabla\rangle\langle\Delta,B'e^\lambda\rangle\;d\mu_{\Delta,\nabla}(\lambda)\\ &= \int\limits_{\partial\,\mathrm{Spec}\,A}\langle B\Delta,e^\lambda\rangle\langle e_\lambda,c'\nabla\rangle\;d\mu_{\Delta,\nabla}(\lambda) = \int\limits_{\partial\,\mathrm{Spec}\,A}\langle x,e_\lambda\rangle\langle e_\lambda,x'\rangle\;d\mu_{\Delta,\nabla}(\lambda).\end{aligned}$$

## 4. Root vectors. Geometric constructions

What can be done in the case of a nonsemisimple algebra $J(0)$? Eigenvectors are definitely unsufficient here.

The finite-dimensional situation suggests that we must complement the eigenvectors by root vectors, which are nontrivial solutions of equations $(A-\lambda\mathbb{I})^k x=0$, $k=1,2,\ldots$. But these vectors can be absent. The main idea of the preceding sections was to continuously extend the action of all related operators to a broader space and try to find the missing vectors there. We show how to construct the desired broader spaces.

In this section we describe the construction in geometric terms.

We have started from a space $V$ and an operator $A$ yielding (DCC). We have managed to construct a broader space $V_-$ and extend the action of $A$ to the space $V_-$ by continuity. Then for every $\lambda\in\mathrm{Spec}\,A$ we have found the related eigenvector $e_\lambda\in V_-$, $Ae_\lambda=\lambda e_\lambda$. Let $E_\lambda$ denote the related eigensubspace; it is one-dimensional due to (DCC). Consider the quotient space $V_-/E_\lambda$. The operator $A$ can naturally be lifted to $V_-/E_\lambda$. Let $A_{1,\lambda}$ be the lifted operator: $A_{1,\lambda}:V_-/E_\lambda\longrightarrow V_-/E_\lambda$.

There are two possibilities:

(i) $\qquad \lambda\in\mathrm{Spec}\,A_{1,\lambda},$

(ii) $\qquad \lambda\notin\mathrm{Spec}\,A_{1,\lambda}.$

If (ii) holds, then we cannot even expect that an eigenvector of the operator $A_{1,\lambda}$ with the eigenvalue $\lambda$ exists, and hence the related associate vector of the initial operator $A$ cannot exist. If (i) holds, then we may expect the existence of the related eigenvector.

Let us study case (i).

**Definition 4.1.** We write $\lambda \in \operatorname{Spec}^1 A$ if $\lambda \in \operatorname{Spec} A_{1,\lambda}$.

We can readily present examples in which there are no eigenvectors in $V_-/E_\lambda$ with eigenvalue $\lambda \in \operatorname{Spec}^1 A$, and thus there are no related associate vectors in $V_-$.

What can we do? As was already shown in Theorem 2.5, we can extend the operator $A_{1,\lambda}$ by continuity to a broader space $(V_-/E_\lambda)_-$ and find the missing eigenvector there. As was explained in Remark 2.7, to construct the space $(V_-/E_\lambda)_-$ we really need only a cyclic vector in

$$E_\lambda^\perp = \{y \in V^+ : \forall x \in E_\lambda \ \langle x, y\rangle = 0\},$$

which is a predual space of $V_-/E_\lambda$ (one can readily see that $(E_\lambda^\perp)' = V_-/E_\lambda$).

Let us first construct everything in terms of the algebra $J(0)$ and then use the rigging. Thus, $(\tau^\nabla)'$ is an isometry of $V_-$ onto $J(0)'$, and $(\tau^\nabla)'E_\lambda = \{\nu\varphi_\lambda : \nu \in \mathbb{C}\}$. Let $(\tau_\lambda^\nabla)'$ be the natural isometry between the quotient space $V_-/E_\lambda$ and the space

$$J(0)'/\{\nu\varphi_\lambda : \nu \in \mathbb{C}\} = (\{\nu\varphi_\lambda : \nu \in \mathbb{C}\}^\perp)' = (\operatorname{Ker} \varphi_\lambda)'.$$

Let $A(0)$ be the operator of multiplication by $A$ in the algebra $J(0)$. In this case the operator given by $(\tau^\nabla)'A{(\tau^\nabla)'}^{-1}$, which corresponds to the operator $A : V_- \to V_-$, is just $(A(0))'$. This can be verified as follows: for any $\Phi \in J(0)'$ and any $B \in J(0)$ we have

$$\begin{aligned}\{[(\tau^\nabla)'A{(\tau^\nabla)'}^{-1}](\Phi)\}(B) \;&=\; \langle {(\tau^\nabla)'}^{-1}(\Phi), A'\tau^\nabla(B)\rangle \;=\; \langle {(\tau^\nabla)'}^{-1}(\Phi), \tau^\nabla(BA)\rangle \\ &=\; \Phi(BA) \;=\; \Phi(A(0)B).\end{aligned}$$

Let $A(1,\lambda)$ be the restriction of the operator $A(0)$ to the subspace $\operatorname{Ker} \varphi_\lambda$. The operator $(\tau_\lambda^\nabla)'A_{1,\lambda}{(\tau_\lambda^\nabla)'}^{-1}$ is obviously dual to the operator $A(1,\lambda)$. In particular, $\operatorname{Spec} A(1,\lambda) = \operatorname{Spec} A_{1,\lambda}$.

Thus, we have a pair of Banach spaces, $(\operatorname{Ker} \varphi_\lambda)'$ and $\operatorname{Ker} \varphi_\lambda$, and a bounded linear operator $A(1,\lambda) : \operatorname{Ker} \varphi_\lambda \to \operatorname{Ker} \varphi_\lambda$.

Now we must define the algebras $\operatorname{Rat}(A(1,\lambda))$ and $R(A(1,\lambda))$, see Definitions 1.1 and 1.2. To this end we must determine the set $\operatorname{Spec} A(1,\lambda)$, see Lemma 4.3 below.

The proof of Lemma 4.3 is based on a natural direct decomposition of the algebra $J(0)$. This decomposition will be repeatedly exploited in this paper.

**Proposition 4.2.** $J(0) = \{\nu\mathbb{I} : \nu \in \mathbb{C}\} \oplus \operatorname{Ker} \varphi_\lambda$.

*Proof.* For any $X \in J(0)$ we have $X = \varphi_\lambda(X)\cdot\mathbb{I} + (X - \varphi_\lambda(X)\cdot\mathbb{I})$. Obviously, the second term belongs to $\operatorname{Ker} \varphi_\lambda$.

**Lemma 4.3.** $\operatorname{Spec} A \supset \operatorname{Spec} A(1,\lambda) \supset (\operatorname{Spec} A) \setminus \{\lambda\}$.

*Proof.* If $\mu \notin \operatorname{Spec} A$, then $(A-\mu\mathbb{I})^{-1} \in J(0)$, and the subspace $\operatorname{Ker} \varphi_\lambda$ is invariant under multiplication by $(A-\mu\mathbb{I})^{-1}$, which implies $\mu \notin \operatorname{Spec} A(1,\lambda)$. Therefore, $\operatorname{Spec} A \supset \operatorname{Spec} A(1,\lambda)$.

If $\mu \neq \lambda$ and $\mu \notin \operatorname{Spec} A(1,\lambda)$, then we can construct a bounded operator in $J(0)$ that is inverse to the multiplication by $(A-\mu\mathbb{I})$, which implies that $\operatorname{Spec} A(1,\lambda) \supset (\operatorname{Spec} A) \setminus \{\lambda\}$.

In order to construct this operator, we must prove that the equation $(A-\mu\mathbb{I})B = C$ has a unique solution $B \in J(0)$ for any $C \in J(0)$.

Decomposing the entries according to Proposition 4.2, we obtain

$$(A-\mu\mathbb{I})(\varphi_\lambda(B)\mathbb{I} + (B-\varphi_\lambda(B)\mathbb{I})) = \varphi_\lambda(C)\mathbb{I} + (C-\varphi_\lambda(C)\mathbb{I})$$

or $\varphi_\lambda(B)(A-\lambda\mathbb{I}) + (\lambda-\mu)\varphi_\lambda(B)\mathbb{I} + (A-\mu\mathbb{I})(B-\varphi_\lambda(B)\mathbb{I}) = \varphi_\lambda(C)\mathbb{I} + (C-\varphi_\lambda(C)\mathbb{I})$. Here the first and the third terms in the left-hand side belong to $\operatorname{Ker}\varphi_\lambda$, and therefore $(\lambda-\mu)\varphi_\lambda(B) = \varphi_\lambda(C)$ and $\varphi_\lambda(B)(A-\lambda\mathbb{I}) + (A(1,\lambda)-\mu\mathbb{I})(B-\varphi_\lambda(B)\mathbb{I}) = C-\varphi_\lambda(C)\mathbb{I}$. Thus,

$$\varphi_\lambda(B) = (\lambda-\mu)^{-1}\varphi_\lambda(C), \qquad B-\varphi_\lambda(B)\mathbb{I} = (A(1,\lambda)-\mu\mathbb{I})^{-1}[C-\varphi_\lambda(C)\mathbb{I}-\varphi_\lambda(B)(A-\lambda\mathbb{I})].$$

Since $\mu \notin \operatorname{Spec} A(1,\lambda)$, it follows that there exists a bounded operator

$$(A(1,\lambda)-\mu\mathbb{I})^{-1} : \operatorname{Ker}\varphi_\lambda \to \operatorname{Ker}\varphi_\lambda.$$

Hence,

$$\begin{aligned} B &= \varphi_\lambda(B)\mathbb{I} + (B-\varphi_\lambda(B)\mathbb{I}) \\ &= (\lambda-\mu)^{-1}\varphi_\lambda(C) + (A(1,\lambda)-\mu\mathbb{I})^{-1}[C-\varphi_\lambda(C)\mathbb{I}-(\lambda-\mu)^{-1}\varphi_\lambda(C)(A-\lambda\mathbb{I})]. \end{aligned}$$

Thus, we have constructed an inverse operator for the multiplication by $(A-\mu\mathbb{I})$ in $J(0)$, and the resulting operator is obviously bounded. This proves the lemma.

**Corollary 4.4.** *If $\lambda$ is a nonisolated point of* $\operatorname{Spec} A$, *then* $\lambda \in \operatorname{Spec} A(1,\lambda)$, *and therefore* $\lambda \in \operatorname{Spec}^1 A$.

**Corollary 4.5.** $\lambda \in \operatorname{Spec}^1 A$ *if and only if* $\operatorname{Spec} A = \operatorname{Spec} A(1,\lambda)$.

In particular, if $\lambda \in \operatorname{Spec}^1 A$, then $\operatorname{Rat}(A(1,\lambda)) = \operatorname{Rat}(A)$.

We are going to find a natural $R(A(1,\lambda))$-cyclic vector in $\operatorname{Ker}\varphi_\lambda$. Consider the ideals $I(k,\lambda) = \{(A-\lambda\mathbb{I})^k B : B \in R(A)\}$, $k = 1,2,\dots,$ of the algebra $R(A)$. By Proposition 3.3, we have $I(1,\lambda) \subset \operatorname{Ker}\varphi_\lambda$, and this inclusion is obviously dense (in the topology induced from $J(0)$). Therefore, the element $(A-\lambda\mathbb{I})$ is a natural cyclic vector for the action of the algebra $R(A(1,\lambda))$ on $\operatorname{Ker}\varphi_\lambda$.

Let $\overline{R}(A(1,\lambda))$ be the closure of $R(A(1,\lambda))$ in the operator norm topology on $\operatorname{Ker}\varphi_\lambda$. This norm is given by the following formula: if $D \in R(A(1,\lambda))$, then

$$\|D\|_{\overline{R}(A(1,\lambda))} = \sup_{B\in R(A)} \frac{\|D\cdot(A-\lambda\mathbb{I})B\|_{J(0)}}{\|(A-\lambda\mathbb{I})B\|_{J(0)}}.$$

Let us imbed the algebra $\overline{R}(A(1,\lambda))$ into the space $\operatorname{Ker}\varphi_\lambda$ with the help of the cyclic vector $(A-\lambda\mathbb{I})$. The imbedding operator $\tau(1,\lambda)$ is defined in a natural way: $\tau(1,\lambda)B = B(A-\lambda\mathbb{I})$. This is a continuous imbedding with dense range, and the range contains the ideal $I(1,\lambda)$.

Equip the lineal $\operatorname{Im}\,\tau(1,\lambda)$ with the norm $\|\cdot\|_{1,\lambda}$ transferred by this imbedding: if $C \in \operatorname{Im}\,\tau(1,\lambda)$, then there exists a unique $D \in \overline{R}(A(1,\lambda))$ such that $C = D(A-\lambda\mathbb{I})$, and we set

$$\|C\|_{1,\lambda} = \|D\|_{\overline{R}(A(1,\lambda))} = \sup_{B\in R(A)} \frac{\|D\cdot(A-\lambda\mathbb{I})B\|_{J(0)}}{\|(A-\lambda\mathbb{I})B\|_{J(0)}} = \sup_{B\in R(A)} \frac{\|CB\|_{J(0)}}{\|(A-\lambda\mathbb{I})B\|_{J(0)}}.$$

Let $\widetilde{J}(1,\lambda)$ denote the Banach space Im $\tau(1,\lambda)$ equipped with the norm $\|\cdot\|_{1,\lambda}$. The space $\widetilde{J}(1,\lambda)$ can also be described as the completion of the ideal $I(1,\lambda)$ with respect to the norm $\|\cdot\|_{1,\lambda}$.

The operator $A(1,\lambda)$ can be restricted to $\widetilde{J}(1,\lambda)$, and by Corollary 2.2 we have

$$\operatorname{Spec}\{A(1,\lambda) : \operatorname{Ker}\varphi_\lambda \to \operatorname{Ker}\varphi_\lambda\} = \operatorname{Spec}\{A(1,\lambda) : \widetilde{J}(1,\lambda) \to \widetilde{J}(1,\lambda)\}.$$

By Theorem 2.5, $\lambda \in \operatorname{Spec} A(1,\lambda)$ if and only if there exists $\varphi_{1,\lambda} \in \widetilde{J}(1,\lambda)'$ such that $\varphi_{1,\lambda} \neq 0$ and $A(1,\lambda)'\varphi_{1,\lambda} = \lambda\cdot\varphi_{1,\lambda}$. The element $\varphi_{1,\lambda}$ is a natural candidate for the first associate vector of the operator $A(0)' = (\tau^\nabla)'A(\tau^\nabla)'^{-1}$, and hence the vector $(\tau^\nabla)'^{-1}\varphi_{1,\lambda}$ is a natural candidate for the first associate vector of the operator $A$.

Unfortunately, $\varphi_{1,\lambda}$ belongs to the space $\widetilde{J}(1,\lambda)'$, which is an extension of the space $(\operatorname{Ker}\varphi_\lambda)' = J(0)'/\{\nu\varphi_\lambda : \nu\in\mathbb{C}\}$. This is natural, since the first associate vector is defined modulo the related eigenvector, but nevertheless we prefer to place it in an extension of the space $J(0)'$.

To this end we again apply the fact that the ideal $\operatorname{Ker}\varphi_\lambda$ is naturally complemented in $J(0)$ (Proposition 4.2): $J(0) = \operatorname{Ker}\varphi_\lambda \oplus \{\nu\mathbb{I} : \nu\in\mathbb{C}\}$. Consider the space $J(1,\lambda) = \widetilde{J}(1,\lambda)\oplus\{\nu\mathbb{I} : \nu\in\mathbb{C}\}$. The space $\widetilde{J}(1,\lambda)$ is continuously and densely imbedded in $\operatorname{Ker}\varphi_\lambda$. Therefore, the space $J(1,\lambda)$ is continuously and densely imbedded in the space $\operatorname{Ker}\varphi_\lambda\oplus\{\nu\mathbb{I} : \nu\in\mathbb{C}\} = J(0)$. This means that the space $J(0)' = (\operatorname{Ker}\varphi_\lambda)'\oplus\{\nu\varphi_\lambda : \nu\in\mathbb{C}\}$ can be continuously and imbedded, with weakly dense image, in the space $J(1,\lambda)' = \widetilde{J}(1,\lambda)'\oplus\{\nu\varphi_\lambda : \nu\in\mathbb{C}\}$. Therefore, the vector $\varphi_{1,\lambda}\in\widetilde{J}(1,\lambda)'$ can be regarded as an element of an extension of the space $J(0)'$. Thus, for every $\lambda\in\operatorname{Spec}^1 A$ we have found the related associate vector $\varphi_{1,\lambda}$ of the operator $A(0)'$ in a suitable extension of the space $J(0)'$. We normalize it by the conditions $\varphi_{1,\lambda}(\mathbb{I}) = 0$ and $\varphi_{1,\lambda}(A-\lambda\mathbb{I}) = 1$. Since $(A(1,\lambda)'-\lambda\mathbb{I})\varphi_{1,\lambda} = 0$, it follows that $\varphi_{1,\lambda}((A(1,\lambda)-\lambda\mathbb{I})^2B) = 0$ for any $B\in R(A)$ (in other words, $\varphi_{1,\lambda}|_{I(2,\lambda)} = 0$).

We can iterate this construction, successively extend our operators to broader spaces, and thus find all possible root vectors in the corresponding spaces. These spaces obviously form an inductive system, and therefore the inductive limit of the obtained spaces will contain all possible root vectors of the operator $A(0)'$. All operators from the algebra $R(A)$ are naturally extendable to this inductive limit.

Let us describe the inductive step in more detail.

Assume that we have already obtained the norms $\|\cdot\|_{p,\lambda}$ defined on the ideals $I(p,\lambda)$ for $p = 1,2,...,k$. Let $\widetilde{J}(p,\lambda)$ denote the completion of the ideal $I(p,\lambda)$ with respect to the norm $\|\cdot\|_{p,\lambda}$. Let $A(p,\lambda)$ denote the restriction of the operator $A(0)$ to $\widetilde{J}(p,\lambda)$.

**Definition 4.6.** We say that $\lambda\in\operatorname{Spec}^p A$ if $\lambda\in\operatorname{Spec} A(p,\lambda)$.

By Corollaries 4.4 and 4.5, nonisolated points of $\operatorname{Spec} A$ belong to $\operatorname{Spec}^p A$ for any $p$, and if $\lambda\in\operatorname{Spec}^p A$, then $\operatorname{Spec} A(p,\lambda) = \operatorname{Spec} A$. In particular, if $\lambda\in\operatorname{Spec}^p A$, then $\operatorname{Rat}(A(p,\lambda)) = \operatorname{Rat}(A)$.

Assume that we have already obtained the elements

$$\varphi_\lambda = \varphi_{0,\lambda},\quad \varphi_{1,\lambda},\,\varphi_{2,\lambda},\ldots,\varphi_{k,\lambda},\qquad \varphi_{p,\lambda}\in\widetilde{J}(p,\lambda)',$$

$$\varphi_{p,\lambda}((A-\lambda\mathbb{I})^p)=1, \qquad \varphi_{p,\lambda}|_{I(p+1,\lambda)}=0, \qquad p=1,2,\dots,k.$$

The vectors $(A-\lambda\mathbb{I})^p\in I(p,\lambda)$ are natural $R(A(p,\lambda))$-cyclic vectors. Let $B(\operatorname{Ker}\varphi_{p,\lambda})$ denote the Banach algebra of all bounded operators in the space $\operatorname{Ker}\varphi_{p,\lambda}$ (the space $\operatorname{Ker}\varphi_{p,\lambda}$ is assumed to be equipped with the norm $\|\cdot\|_{p-1,\lambda}$). Let $\overline{R}(A(p,\lambda))$ denote the closure of the algebra $R(A(p,\lambda))$ in the Banach algebra $B(\operatorname{Ker}\varphi_{p,\lambda})$. As above, we obtain dense inclusions

$$\tau(p,\lambda):\ \overline{R}(A(p,\lambda))\to\operatorname{Ker}\varphi_{p-1,\lambda}\,, \qquad \tau(p,\lambda)(B)=B(A-\lambda\mathbb{I})^p.$$

Let $A(k+1,\lambda)$ denote the restriction of the operator $A(k,\lambda):\ \widetilde{J}(k,\lambda)\to\widetilde{J}(k,\lambda)$ to the subspace $\operatorname{Ker}\varphi_{k,\lambda}$. The ideal $I(k+1,\lambda)$ is obviously dense in $\operatorname{Ker}\varphi_{k,\lambda}$. Let us construct a dense injection $\tau(k+1,\lambda)$ with the help of the $R(A(k+1,\lambda))$-cyclic vector $(A-\lambda\mathbb{I})^{k+1}$ of the form

$$\tau(k+1,\lambda):\ \overline{R}(A(k,\lambda))\to\operatorname{Ker}\varphi_{k,\lambda}\,, \qquad \tau(k+1,\lambda)(B)=B(A-\lambda\mathbb{I})^{k+1}.$$

Transfer the norm from the algebra $\overline{R}(A(k+1,\lambda))$ to the lineal $\operatorname{Im}\tau(k+1,\lambda)$ with the help of the mapping $\tau(k+1,\lambda)$. Let $\|\cdot\|_{k+1,\lambda}$ denote the transferred norm and, let $\widetilde{J}(k+1,\lambda)$ denote the resulting Banach space.

On restricting the operator $A(k+1,\lambda)$ to $\widetilde{J}(k+1,\lambda)$, we obtain the operator $A(k+1,\lambda):\ \widetilde{J}(k+1,\lambda)\to\widetilde{J}(k+1,\lambda)$. By Corollary 2.2, we have

$$\operatorname{Spec}\{A(k{+}1,\lambda):\ \widetilde{J}(k{+}1,\lambda)\to\widetilde{J}(k{+}1,\lambda)\}=\operatorname{Spec}\{A(k{+}1,\lambda):\operatorname{Ker}\varphi_{k,\lambda}\to\operatorname{Ker}\varphi_{k,\lambda}\},$$

and if $\lambda\in\operatorname{Spec}^{k+1}A$, then $\lambda\in\operatorname{Spec}A(k+1,\lambda)$; moreover, we can find a vector $\varphi_{k+1,\lambda}\in\widetilde{J}(k+1,\lambda)'$ such that $\varphi_{k+1,\lambda}((A-\lambda\mathbb{I})^{k+1})=1$ and $\varphi_{k+1,\lambda}|_{I(k+2,\lambda)}=0$. Now the inductive step is completely described.

The ideals $I(k,\lambda)$ are naturally complemented in $R(A)$, and the related decompositions are given by the following formulas, where the first term in the right-hand side obviously belongs to $I(k,\lambda)$ :

$$f(A)=\Big(f(A)-\sum_{i=0}^{k-1}(i!)^{-1}f^{(i)}(\lambda)(A-\lambda\mathbb{I})^i\Big)+\Big(\sum_{i=0}^{k-1}(i!)^{-1}f^{(i)}(\lambda)(A-\lambda\mathbb{I})^i\Big).$$

Introduce the $k$-dimensional subspaces

$$T(k,\lambda)=\Big\{\sum_{i=0}^{k-1}c_i(A-\lambda\mathbb{I})^i:\ c_i\in\mathbb{C}\Big\}.$$

Then we have $R(A)=I(k,\lambda)\oplus T(k,\lambda)$. Consider the spaces $J(k,\lambda)=\widetilde{J}(k,\lambda)\oplus T(k,\lambda)$. Then $J(k,\lambda)'=\widetilde{J}(k,\lambda)'\oplus T(k,\lambda)'$. Since $\varphi_{k,\lambda}\in\widetilde{J}(k,\lambda)'$, we have $\varphi_{k,\lambda}|_{T(k,\lambda)}=0$, and regarding the vectors $\varphi_{k,\lambda}$ as elements of $J(k,\lambda)'$ (and therefore as elements of an extension of $J(0)'$) we see that $\varphi_{k,\lambda}((A-\lambda\mathbb{I})^p)=\delta(k-p)$. This is equivalent to the relation $(A(0)'-\lambda\mathbb{I})^p\varphi_{k,\lambda}=\varphi_{k-p,\lambda}$, where we certainly assume that $\varphi_{p,\lambda}=0$ for $p\le-1$.

Obviously, $J(0)\supset J(1,\lambda)\supset J(2,\lambda)\supset\cdots$, and all inclusions are dense and continuous. Consider the intersection

$$J_\infty(A)=\bigcap_{k\ge0,\ \lambda\in\operatorname{Spec}^k A}J(k,\lambda).$$

It contains $R(A)$ and can be equipped with the natural locally convex topology of the projective limit. We can find all possible root vectors in the dual space of $J_\infty(A)$, which we denote by $J^\infty(A)$.

We conclude this section with the following assertion.

**Proposition 4.7.** *Let $B$ and $C$ belong to the algebra $J_\infty(A)$. Then*

$$\varphi_{k,\lambda}(BC) = \sum_{s_1+s_2=k} \varphi_{s_1,\lambda}(B)\varphi_{s_2,\lambda}(C).$$

*Proof.* Since $B, C \in J_\infty(A)$, we have $B, C \in J(k+1,\lambda)$. Let $B = B_k + b_k$ and $C = C_k + c_k$ be the standard decompositions, where $B_k, C_k \in \widetilde{J}(k+1,\lambda)$ and $b_k, c_k \in T(k+1,\lambda)$. Then

$$b_k = \sum_{0\le s\le k} \beta_s (A-\lambda\mathbb{I})^s, \qquad c_k = \sum_{0\le s\le k} \gamma_s (A-\lambda\mathbb{I})^s.$$

Let us take into account the fact that $\varphi_{s,\lambda}((A-\lambda\mathbb{I})^t) = \delta(s-t)$. Then for any $s$ between 0 and $k+1$ we obtain

$$\varphi_{s,\lambda}(B) = \varphi_{s,\lambda}(B_k+b_k) = \varphi_{s,\lambda}(B_k) + \varphi_{s,\lambda}(b_k) = 0 + \sum_{0\le t\le k} \beta_t \varphi_{s,\lambda}((A-\lambda\mathbb{I})^t) = \beta_s.$$

Similarly, $\varphi_{s,\lambda}(C) = \gamma_s$. Therefore,

$$\varphi_{k,\lambda}(BC) = \varphi_{k,\lambda}\Big(\sum_{s_1+s_2\le k} \varphi_{s_1,\lambda}(B)\ \varphi_{s_2,\lambda}(C)(A-\lambda\mathbb{I})^{s_1+s_2} + (\text{an element of } \widetilde{J}(k+1,\lambda))\Big)$$
$$= \sum_{s_1+s_2=k} \varphi_{s_1,\lambda}(B)\ \varphi_{s_2,\lambda}(C).$$

In the next part of the paper (Sections 5–7) we describe the entire construction in analytic terms, which permits us to continue the study of the Jordan decomposition.

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