BIRKHOFF'S THEOREM AND CONVEX HULLS OF COXETER GROUPS

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Abstract. We formulate and partially prove a general conjecture regarding the facial structure of convex hulls of finite irreducible Coxeter groups.

1. Introduction

Let G be a finite irreducible Coxeter group naturally acting on a finite dimensional real Euclidean space V. So, G is a finite subset in the space $\operatorname{End} V$ of linear operators in V. Consider $\operatorname{conv} G$ — the convex hull of G. This is a convex polyhedron in the linear space $\operatorname{End} V$. We are interested in its facial structure and especially in calculation of normals to its faces of maximal dimension. These (properly scaled) normals are naturally identified with elements of $\operatorname{Extr}(\operatorname{conv} G)^\circ$ — the set of extreme elements of the polar set $(\operatorname{conv} G)^\circ$.

The well known Birkhoff's Theorem [2] concerning the extreme points of the set of bistochastic matrices is a result of exactly this type, giving an answer in the case of the group A_n — see Section 4 below.

Each weight ω of the group G is associated with a vertex $\pi(\omega)$ of the Coxeter graph $\Gamma(G)$. Let E_G denote the set of **extremal weights** of the group G, i.e., those associated with the **end vertices** of the Coxeter graph $\Gamma(G)$.

Put $m_G(x, y) = \max\{\langle gx, y\rangle : g \in G\}$. Let

$$\mathcal{B}_G = \{(\omega \otimes \tau)/m_G(\omega, \tau) : \omega, \tau \in E_G, \pi(\omega) \neq \pi(\tau)\}.$$

We call the elements of $\mathcal{B}_G$ the **Birkhoff tensors.**

The importance of Birkhoff tensors for our problem is apparent because of the following result (see Theorem 3.4 below): $\mathcal{B}_G = (\operatorname{Extr}(\operatorname{conv} G)^\circ) \bigcap (\text{ rank 1 tensors })$.

The following conjecture was first proposed in 1979 by Veronica Zobin [8] and later elaborated by the last author:

Conjecture 1.1. *(a) If the Coxeter graph $\Gamma(G)$ is non-branching then*

$$\mathcal{B}_G = \operatorname{Extr}(\operatorname{conv} G)^\circ.$$

(b) If the Coxeter graph $\Gamma(G)$ is branching then

$$\mathcal{B}_G \subsetneq \operatorname{Extr}(\operatorname{conv} G)^\circ.$$

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We show that Part (a) of this Conjecture is true for all **infinite families** of Coxeter groups with non-branching graphs (i.e., for the families $A_n, B_n, I_2(n)$); in the first two cases we show that this assertion is essentially a reformulation of the Birkhoff's Theorem. So, to confirm the Conjecture for all groups with non-branching graphs, one has to consider the remaining exceptional groups F_4, H_3, H_4. The cases of F_4 and H_3 were recently verified by computer calculations, carried out by J. Brandman, J. Fowler, B. Lins, and the last two authors. The details will appear elsewhere.

We prove Part (b) of this Conjecture for **all** Coxeter groups with branching graphs by presenting an essentially unique tensor of rank 3 belonging to the set $\operatorname{Extr}(\operatorname{conv} D_4)^{\circ}$, and then reducing the general case to this one. Our D_4-example was produced by a computer calculation.

So, as of September 2001, the only remaining unsettled case in the Conjecture is the group H_4.

The above conjecture naturally appeared in the theory of operator interpolation in spaces with given symmetries — see [7, 6]. Consider the convex set $\operatorname{env} G = (\mathcal{B}_G)^{\circ}$. It is the semigroup of all linear operators in V which transform every G-invariant convex set into itself. Certainly, $\operatorname{conv} G \subset \operatorname{env} G$. If these two sets coincide then there are no non-obvious operators contracting all G-invariant convex sets. So this case is not interesting from the point of view of operator interpolation. The opposite case is much more interesting — there are nontrivial operators that can be interpolated.

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2. A brief review of Coxeter groups

Let us now address several facts concerning the theory of Coxeter groups. For greater detail, consult [1], [3], or [4]. Let $G \subset \operatorname{End} V$ be a group. Then G is a **Coxeter group** if it is finite, generated by orthogonal reflections across hyperplanes (containing the origin), and has no nontrivial fixed points (i.e., $gx = x$ for all $g \in G$ implies $x = 0$).

Fix a Weyl chamber C. For $x \in V$ let x^* denote the only common point of the G-orbit of x and C. Consider $m_G(x, y) = \max_{g \in G} \langle gx, y\rangle$. It is known (see [7]) that $m_G(x, y) = \langle x^*, y^*\rangle \geq 0$. In fact, one can easily show that $m_G(x, y) > 0$ for irreducible Coxeter groups, provided x, y are both nonzero.

For every wall W_i of C, let n_i be the related **fundamental root**. We choose all roots to be unit vectors. Let ω_j be the related **fundamental weight**, so that $\langle n_i, \omega_j\rangle = c_j\delta_{ij},\ c_j > 0$. The exact value of c_j (or, in other terms, the normalization of ω_j) is not important for our purposes.

An **end vertex** of the Coxeter graph $\Gamma(G)$ is any vertex connected to only one other vertex. A Coxeter graph is **branching** if it contains a vertex connected to at least three other vertices. Otherwise, the graph is **non-branching**.

For $a \in C$ let

$$\operatorname{Stab}_G a = \{g \in G : ga = a\}$$

It is well known (see, e.g., [3]) that this subgroup is generated by reflections across the walls W_i of C, containing a. So, this subgroup is not a Coxeter group since it has nontrivial fixed vectors. Restrict the action to its invariant subspace $V_a = (\bigcap_{a \in W_i} W_i)^{\perp}$. Thus, the nontrivial fixed vectors are cut off, and we get a Coxeter group $\mathrm{Stab}_G\, a|_{V_a}$ acting on this subspace V_a. Its Coxeter graph of can be computed as follows (see [7]):

$$\Gamma(\mathrm{Stab}_G\, a|_{V_a}) = \Gamma(G) \setminus \{\pi_i \in \Gamma(G) : a \notin W_i\}$$

The latter means that all the vertices from $\{\pi_i \in \Gamma(G) : a \notin W_i\}$ are erased, as well as all adjacent edges. So, if ω is a fundamental weight then $V_\omega = \omega^\perp$ and $\Gamma(\mathrm{Stab}_G\, \omega|_{\omega^\perp}) = \Gamma(G) \setminus \{\pi(\omega)\}$. If ω is an extremal fundamental weight then the group $\mathrm{Stab}_G\, \omega$ acts irreducibly on $\omega^\perp$, since the graph $\Gamma(G) \setminus \{\pi(\omega)\}$ is connected.

There exist four infinite families of irreducible Coxeter groups plus six exceptional groups. We are going to need a description of these families in Sections 4–6.

Define Perm_n as the group of linear operators acting on $\mathbf{R}^n$ by permutations of the canonical basis.

Groups $\boldsymbol{A_n}$.

Let $\{e_1, e_2, \ldots, e_{n+1}\}$ be the standard basis of $\mathbf{R}^{n+1}$. Consider the group Perm_{n+1}. Then $d = \sum_{i=1}^{n+1} e_i$ is a fixed vector, and therefore the subspace $d^\perp$ is invariant under the action of Perm_{n+1}.

Definition 2.1. $A_n = \{T|_{d^\perp} : T \in \mathrm{Perm}_{n+1}\}$.

Vectors $\omega_1 = e_1 - d/(n+1),\ \omega_n = d/(n+1) - e_{n+1}$ are extremal fundamental weights.

Note for future reference that ω_1 is the orthogonal projection of the vector e_1 onto the subspace $d^\perp$. Also, A_n does not contain $-\mathbf{I}$ (where $\mathbf{I}$ is the identity operator), and $\omega_n \in \mathrm{Orb}_{A_n}(-\omega_1)$.

Groups $\boldsymbol{B_n}$.

Definition 2.2. *B_n is the group of linear operators acting on $\mathbf{R}^n$ by taking e_i to $s(i)e_{\sigma(i)}$, where σ is a permutation of $\{1, \ldots, n\}$, and $s(i) = \pm 1$ for $1 \le i \le n$.*

The vectors $\omega_1 = e_1,\ \omega_n = e_1 + \cdots + e_n$ are extremal fundamental weights.

Groups $\boldsymbol{I_2(n)}$.

Definition 2.3. *For $n \ge 3$, let $I_2(n)$ be the group of operators acting on $\mathbf{R}^2$ generated by reflections across the line $y = 0$ and the line $y = \tan(\pi/n)x$.*

Groups $\boldsymbol{D_n}$.

Definition 2.4. $D_n = \{T \in B_n : T$ *performs an* **even** *number of sign changes*$\}$.

3. Convex Geometry and Irreducible Coxeter Groups

Space $\mathrm{End}\, V$ becomes Euclidean if equipped with the scalar product $(T, S) = \mathrm{trace}\,(TS^*)$. Identify $x \otimes y$ with the rank 1 operator $z \to x\langle z, y\rangle$. One can easily check that $(x \otimes y)^* = y \otimes x$, $\mathrm{trace}\,(x \otimes y) = \langle x, y\rangle$, $(x \otimes y)(w \otimes t) = (x \otimes t)\langle y, w\rangle$.

As usual, if we have a real Euclidean space W with a scalar product $(.,.)$ then for a subset $U \subset W$ we consider its **polar set** $U^\circ = \{z \in W : \forall\, x \in U \quad (x, z) \le 1\}$. The set U° is a closed convex subset of W, containing 0. One can easily verify that $U^\circ = (\mathrm{conv}\, U)^\circ$. By the Bipolar Theorem, $(U^\circ)^\circ = \overline{\mathrm{conv}}\,(U \cup \{0\})$. So if $0 \in \mathrm{conv}\, G$ then $\mathrm{conv}\, G = ((\mathrm{conv}\, G)^\circ)^\circ$ (we may omit the closure since $\mathrm{conv}\, G$ is a closed polyhedron). We show that 0 is an **interior** point of $\mathrm{conv}\, G$, see Lemma

3.2. This implies that the set $(\operatorname{conv} G)^\circ$ is compact and therefore, by the Krein-Milman Theorem, this set is a closed convex hull of its extreme points. So the set $\operatorname{Extr}(\operatorname{conv} G)^\circ$ provides a nice description of the set $\operatorname{conv} G$:

$$\operatorname{conv} G = (\operatorname{Extr}(\operatorname{conv} G)^\circ)^\circ = \{T \in \operatorname{End} V : (T, b) \le 1 \quad \forall\, b \in \operatorname{Extr}(\operatorname{conv} G)^\circ\}.$$

This formula and the definition of extreme points show that the elements of the set $\operatorname{Extr}(\operatorname{conv}(G)^\circ$ are properly scaled normals to faces of the polyhedron $\operatorname{conv} G$.

The following lemma can be deduced from the Burnside Theorem, but we prefer to give a simple direct proof, especially because its idea is also used in the proof of Theorem 3.4 below.

Lemma 3.1. *Let G be an irreducible Coxeter group. Then the set G spans the whole space* $\operatorname{End} V$.

Proof. Fix a Weyl chamber C. As before let $n_i,\ i = 1, 2, ..., \dim V$, denote the roots (the **unit normals** to the walls of C), associated with the vertices π_i of the Coxeter graph $\Gamma(G)$. These roots form a basis of V. Group G is generated by the reflections $R_i = \mathbf{I} - 2n_i \otimes n_i,\ i = 1, 2, ..., \dim V$. Since $\mathbf{I} \in G$ then all operators $n_i \otimes n_i$ are in span G. Considering the products $R_i R_j = \mathbf{I} - 2n_i \otimes n_i - 2n_j \otimes n_j + 4\langle n_i, n_j\rangle n_i \otimes n_j$ such that the vertices π_i, π_j are connected by an edge (and therefore $\langle n_i, n_j\rangle \neq 0$), we prove that all such operators $n_i \otimes n_j$ are in span G. Now choose three vertices π_i, π_j, π_k such that the second one is connected by edges to the first and the third ones. Considering the product $R_i R_j R_k \in G$ and using the previous remarks, we show that $n_i \otimes n_k \in \operatorname{span} G$. Repeating the same trick, we show that all operators $n_i \otimes n_j$ are in span G, provided the vertices π_i, π_j can be connected by a simple path in $\Gamma(G)$. Since the Coxeter graph of an irreducible group is connected, the Lemma is proven. ∎

Lemma 3.2. *Let G be an irreducible Coxeter group. Then* 0 *is an interior point of the set* $\operatorname{conv} G$.

Proof. Consider the arithmetic mean av_G of the elements of G. The group G obviously fixes every element in the range of av_G, but this irreducible group cannot have nonzero fixed vectors, therefore $av_G = 0$. So, $0 = av_G \in \operatorname{conv} G$. Assuming that 0 is **not** an interior point of conv G, we find a nonzero operator $b \in \operatorname{End} V$ such that $(g, b) \le 0$ for all $g \in G$. Therefore either $G \subset \{a \in \operatorname{End} V : (a, b) = 0\}$, or $(av_G, b) < 0$. The first is impossible because G spans the space End V, the second is impossible because $av_G = 0$. ∎

This result implies that the set $G^\circ = (\operatorname{conv} G)^\circ$ is compact and therefore $G^\circ = \operatorname{conv} \operatorname{Extr} G^\circ$.

Corollary 3.3. *Let G be an irreducible Coxeter group. Then $0 \in \operatorname{conv}(\mathcal{B}_G)$.*

Theorem 3.4. $\mathcal{B}_G = (\operatorname{Extr} G^\circ) \cap ($ *rank 1 tensors* $)$

Proof. According to [7], $\mathcal{B}_G = \operatorname{Extr} \operatorname{conv}(G^\circ \cap ($ rank 1 tensors $))$. Let us prove that $\mathcal{B}_G \subset \operatorname{Extr} G^\circ$. This will obviously imply the assertion of the Theorem.

Choose two extremal fundamental weights ω, τ belonging to a Weyl chamber C, such that $\pi(\omega) \neq \pi(\tau)$. It suffices to show that $(\omega \otimes \tau)/m_G(\omega, \tau) \in \operatorname{Extr} G^\circ$. Consider the set $\mathcal{M} = \{g \in G : (g, \omega \otimes \tau) = \langle g\tau, \omega\rangle = m_G(\tau, \omega) = m_G(\omega, \tau)\}$. Since for any $g \in G$ we have $(g, \omega \otimes \tau) = \langle g\tau, \omega\rangle \le m_G(\omega, \tau)$ then $\operatorname{conv} \mathcal{M}$ is a face of

$\operatorname{conv} G$ and all we need to show is that its dimension is maximal, i.e., to prove that $\mathcal{M}$ spans End V.

Define $\mathcal{P} = \{hg : h \in \operatorname{Stab}_G(\omega), g \in \operatorname{Stab}_G(\tau)\}$. Obviously, $\mathcal{P} \subset \mathcal{M}$.

Let $n_i,\ 1 \le i \le N = \dim V$, denote the roots associated with the chamber C, we assume that all roots are of unit length. Let $\omega_i,\ 1 \le i \le N$, denote the related fundamental weights, we assume that $\tau = \omega_1,\ \omega = \omega_N$. Let $R_j = \mathbf{I} - 2n_j \otimes n_j$ be the corresponding reflections. Recall that $\operatorname{Stab}_G(\omega_i)$ is generated by $\{R_j : j \ne i\}$.

Obviously, $\mathbf{I} \in \operatorname{Stab}_G(\omega_1) \bigcap \operatorname{Stab}_G(\omega_N)$. Also, note $R_j \in \operatorname{Stab}_G(\omega_1)$ for all $1 < j \le N$, and $R_j \in \operatorname{Stab}_G(\omega_N)$ for all $1 \le j < N$. Thus, for all $1 < j \le N,\ n_j \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_1)$. Similarly, for all $1 \le j < N,\ n_j \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_N)$.

Choose any i, j such that $1 < i, j \le N$. Let $\pi_i = \pi_{k_1}, \pi_{k_2}, \dots, \pi_{k_r} = \pi_j$ be the sequence of vertices along a simple path in $\Gamma(G)$, connecting π_i to π_j. Such a path exists since $\Gamma(G)$ is connected. For all $1 \le l \le r$, we see that $k_l \ne 1$, so $n_{k_l} \otimes n_{k_l} \in \operatorname{span} \operatorname{Stab}_G(\omega_1)$. Then the product $(n_{k_1} \otimes n_{k_1})(n_{k_2} \otimes n_{k_2}) \dots (n_{k_r} \otimes n_{k_r})$ is also in $\operatorname{span} \operatorname{Stab}_G(\omega_1)$. Since

$$\begin{aligned}&(n_{k_1} \otimes n_{k_1})(n_{k_2} \otimes n_{k_2}) \dots (n_{k_r} \otimes n_{k_r})\\ &\qquad = \langle n_{k_1}, n_{k_2}\rangle \langle n_{k_2}, n_{k_3}\rangle \dots \langle n_{k_{r-1}}, n_{k_r}\rangle (n_{k_1} \otimes n_{k_r})\end{aligned}$$

and for any $1 \le l < r$, $\langle n_{k_l}, n_{k_{l+1}}\rangle \ne 0$ (the vertices π_{k_l} and $\pi_{k_{l+1}}$ are connected in $\Gamma(G)$), then $n_{k_1} \otimes n_{k_r} = n_i \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_1)$ for all $1 < i, j \le N$. Repeating the same argument for $\operatorname{Stab}_G(\omega_N)$ yields $n_i \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_N)$ for all $1 \le i, j < N$.

Choose π_m adjacent to π_1. Now $(n_1 \otimes n_1)(n_m \otimes n_N) \in \operatorname{span}(\mathcal{P})$. Since $\langle n_1, n_m\rangle \ne 0$, $n_1 \otimes n_N \in \operatorname{span}(\mathcal{P})$.

To show $n_N \otimes n_1 \in \operatorname{span}(\mathcal{P})$ requires a bit more refined argument. Since the system $\{n_i : 1 \le i \le N\}$ is a basis in the space V, and the system $\{\omega_i : 1 \le i \le N\}$ is biorthogonal to this basis, then one can easily show that

$$\mathbf{I} = \sum_{i,j=1}^{N} \langle \omega_i, \omega_j\rangle n_j \otimes n_i$$

Notice $\langle \omega_i, \omega_j\rangle \ne 0$ (in fact, > 0) for any $1 \le i, j \le N$, because ω_i and ω_j are in the same Weyl chamber, and G is irreducible. For all $(i, j) \ne (N, 1)$, $n_i \otimes n_j \in \operatorname{span}(\mathcal{P})$, and $\mathbf{I} \in \operatorname{span}(\mathcal{P})$, so

$$\mathbf{I} - \sum_{\substack{1 \le i,j \le N\\ (i,j) \ne (N,1)}} \langle \omega_i, \omega_j\rangle n_i \otimes n_j \quad = \quad \langle \omega_n, \omega_1\rangle n_N \otimes n_1$$

is in $\operatorname{span}(\mathcal{P})$. Therefore, $n_N \otimes n_1 \in \operatorname{span}(\mathcal{P})$.

Thus, for all $1 \le i, j \le N,\ n_i \otimes n_j \in \operatorname{span} \mathcal{P}$. Since $\{n_i \otimes n_j : 1 \le i, j \le N\}$ form a basis for End V, and since $\mathcal{P} \subset \mathcal{M}$ we see that $\mathcal{M}$ spans End V as required. ∎

Corollary 3.5. *The following are equivalent:*

1. ${\mathcal{B}_G}^{\circ} \subset \operatorname{conv} G$.
2. ${\mathcal{B}_G}^{\circ} = \operatorname{conv} G$.
3. $\operatorname{Extr}(G^{\circ}) = \mathcal{B}_G$.
4. $\operatorname{Extr}(G^{\circ}) \subset \mathcal{B}_G$.

4. Proof of the Conjecture for the groups A_n and B_n

Definition 4.1. *Let $T = (t_{ij})$ be an $n \times n$ matrix. It is called* **bistochastic** *if its entries are non-negative and*

$$\text{for every } j,\ 1 \leq j \leq n, \qquad \sum_{i=1}^{n} t_{ij} = 1, \qquad \sum_{i=1}^{n} t_{ji} = 1$$

The set of all bistochastic $n \times n$ matrices is denoted Ω_n.

Theorem 4.2 (Birkhoff, [2])**.** Extr $\Omega_n = \mathrm{Perm}_n$.

Birkhoff's Theorem can be reformulated as follows:

Theorem 4.3. Extr $A_n^\circ = \mathcal{B}_{A_n}$.

Proof. Recall that $d = \sum_{i=1}^{n+1} e_i$. Definition 4.1 means that $T \in \Omega_{n+1}$ if and only if $Td = d$, $T^* d = d$ and T transforms the positive ortant of $\mathbf{R}^{n+1}$ into itself. So, $d^\perp$ is invariant under $T \in \Omega_{n+1}$. Therefore T transforms the intersection of the positive ortant with the affine hyperplane

$$\frac{1}{n+1} d + d^\perp$$

into itself. It is easy to see that this intersection is precisely $\mathrm{conv}\,\mathrm{Orb}_{\,\mathrm{Perm}_{n+1}}\, e_1$. Therefore T also transforms the set S – the orthogonal projection of this intersection onto the subspace $d^\perp$ – into itself. Since $\omega_1 = \mathrm{proj}_{\,d^\perp}\, e_1$ (see the description of A_n in Section 2) then

$$S = \mathrm{proj}_{\,d^\perp}\, \mathrm{conv}\,\mathrm{Orb}_{\,\mathrm{Perm}_{n+1}}\, e_1 = \mathrm{conv}\,\mathrm{Orb}_{\,A_n}\, \mathrm{proj}_{\,d^\perp}\, e_1 = \mathrm{conv}\,\mathrm{Orb}_{\,A_n}\, \omega_1$$

It is known (see [7]) that

$$\mathrm{Extr}\,(\mathrm{Orb}_{\,A_n}\, \omega_1)^\circ = \frac{1}{m_G(\omega_1, \omega_n)}\, \mathrm{Orb}_{\,A_n}\, \omega_n.$$

So we conclude that $TS \subset S$ if and only if $(T, h\omega_n \otimes g\omega_1) = \langle Tg\omega_1, h\omega_n\rangle \leq m_G(\omega_1, \omega_n)$ for all $g, h \in A_n$. Since $\omega_1 \in \mathrm{Orb}_{\,A_n}(-\omega_n)$ and $\omega_n \in \mathrm{Orb}_{\,A_n}(-\omega_1)$, the sets $\{h\omega_n \otimes g\omega_1 : g, h \in A_n\}$ and $\{g\omega_1 \otimes h\omega_n : g, h \in A_n\}$ coincide. Therefore $T \in \Omega_{n+1}$ if and only if $Td = d, Td^\perp \subset d^\perp$ and $T|_{d^\perp} \in (\mathcal{B}_{A_n})^\circ$. This means that $\mathrm{Extr}\,(\Omega_{n+1}|_{d^\perp})^\circ \subset \mathcal{B}_{A_n}$. By the Birkhoff's Theorem, Extr $\Omega_{n+1} = \mathrm{Perm}_{n+1}$, so $(\Omega_{n+1})^\circ = (\mathrm{Perm}_{n+1})^\circ$ and, since both Ω_{n+1} and Perm_{n+1} leave $d^\perp$ invariant, we get

$$\mathrm{Extr}\ A_n^\circ = \mathrm{Extr}\,(\mathrm{Perm}_{n+1}\,|_{d^\perp})^\circ = \mathrm{Extr}\,(\Omega_{n+1}|_{d^\perp})^\circ \subset \mathcal{B}_{A_n}.$$

According to Lemma 3.5, this proves the result. ∎

Definition 4.4. *A matrix $(a_{ij}) \in M_n(\mathbf{R})$ is called* **absolutely bistochastic** *if*

$$\text{for every } j,\ 1 \leq j \leq n, \qquad \sum_{i=1}^{n} |a_{ij}| \leq 1, \qquad \sum_{i=1}^{n} |a_{ji}| \leq 1.$$

Let $\mho_n$ be the set of all absolutely bistochastic $n \times n$ matrices.

The next lemma follows from the Birkhoff's Theorem — see, for example, [5].

Lemma 4.5. $B_n = \mathrm{Extr}\,(\mho_n)$

The desired description is now (almost) immediate.

Theorem 4.6. Extr $B_n^\circ = \mathcal{B}_{B_n}$.

Proof. By Corollary 3.5, it suffices to prove $(\mathcal{B}_{B_n})^\circ \subset \mathrm{conv}(B_n)$. Yet, by Lemma 4.5, this statement is equivalent to $(\mathcal{B}_{B_n})^\circ \subset \mho_n$. Let $A = (a_{ij}) \in (\mathcal{B}_{B_n})^\circ$. Let $q = \sum_{j=1}^n \varepsilon_j e_j, \quad \varepsilon_j = \pm 1$. All such q form the B_n-orbit of the extremal fundamental weight ω_n. Then $(A, q \otimes e_i) = \langle Ae_i, q\rangle \leq 1$ for all $q \in Q, 1 \leq i \leq n$. This is equivalent to $\sum_{j=1}^n \varepsilon_j a_{ij} \leq 1$ for all $\varepsilon_j = \pm 1$, $1 \leq i \leq n$, or $\sum_{j=1}^n |a_{ij}| \leq 1$. Similarly, using $(A, e_i \otimes q)$, deduce $\sum_{i=1}^n |a_{ij}| \leq 1$. So $A \in \mho_n$. ∎

5. A Description of Extreme Points of $I_2(n)^\circ$.

Recall that the group $I_2(n)$ is the dihedral group acting on $\mathbf{R}^2$. Let Rot (θ) be the linear operator performing counter-clockwise rotation by the angle θ. Let Refl (0) be the orthogonal reflection across the x-axis, let Refl $(\theta) =$ Rot (θ) Refl (0) Rot $(-\theta)$ be the orthogonal reflection across the line at an angle θ from the x-axis in the counter-clockwise direction. Group $I_2(n)$ is generated by reflections $R_1 =$ Refl (0) and $R_2 =$ Refl (π/n).

Lemma 5.1. $I_2(n) = \{\mathrm{Rot}\,(2\pi k/n) : 0 \leq k < n\} \bigcup \{\mathrm{Refl}\,(\pi k/n) : 0 \leq k < n\}$.

Lemma 5.2. *Let $\varphi : V \to V$ be an invertible linear operator. Let $U \subset V$ be a convex set with $\varphi(U) = U$. Then $\varphi(\mathrm{Extr}\, U) = \mathrm{Extr}\, U$, $\varphi^*(U^\circ) = U^\circ$, and $\varphi^*(\mathrm{Extr}\, U^\circ) = \mathrm{Extr}\, U^\circ$.*

Corollary 5.3. *Let G be a group of orthogonal transformations acting on V. If $A \in \mathrm{Extr}\, G^\circ$, then $A^* \in \mathrm{Extr}\, G^\circ$.*

Corollary 5.4. *Let G be a group of orthogonal transformations acting on V. Then for every $g, h \in G$, if $A \in \mathrm{Extr}\, G^\circ$, then $gAh \in \mathrm{Extr}\, G^\circ$.*

Corollary 5.5. *Let $P = \mathrm{Refl}\,(\pi/2n)$. Define $\phi(A) = PAP^{-1} = PAP$. Then ϕ fixes the identity and sends R_2 to R_1. Moreover, $\phi(\mathrm{Extr}\, I_2(n)^\circ)) = \mathrm{Extr}\, I_2(n)^\circ$.*

Lemma 5.6. *Any face of* conv $I_2(n)$ *containing the identity operator can contain at most one additional rotation.*

Proof. Let $S = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \mathrm{Extr}\, I_2(n)^\circ$ be the normal to a face F, containing the identity. By Corollary 5.3, $S^* \in \mathrm{Extr}\, I_2(n)^\circ$. Moreover, the transposition of a rotation is a rotation, and the transposition of a reflection is a reflection. Hence, without loss of generality, assume $y \geq x$. Also, assume there are two other rotations in $I_2(n)$, $g_{k_i} = \mathrm{Rot}\,(2\pi k_i/n)$ for $i = 1, 2$, and $0 \leq k_i < n$, belonging to F. Since $I \in F$, $(I, S) = 1$, so $w + z = 1$. Also, $(g_{k_i}, S) = (w + z)\cos(2\pi k_i/n) + (y - x)\sin(2\pi k_i/n) = 1$. Suppose $\sin(2\pi k_i/n) = 0$. Then $\cos(2\pi k_i/n) = \pm 1$, which implies $g_{k_i} = \mathbf{I}$ or $-\mathbf{I}$. However, g_{k_i} is not $\mathbf{I}$ by assumption. If $g_{k_i} = -\mathbf{I}$ then $(S, g_{k_i}) = 1$, so $(-w) + (-z) = 1$ — a contradiction. Simplifying the above equation for (S, g_{k_i}), we get

$$y - x = \frac{1 - \cos(2\pi k_i/n)}{\sin(2\pi k_i/n)} = \frac{2\sin^2(\pi k_i/n)}{2\sin(\pi k_i/n)\cos(\pi k_i/n)} = \tan\left(\frac{\pi k_i}{n}\right).$$

Hence $\tan(\pi k_1/n) = \tan(\pi k_2/n)$. Then $\sin(\pi k_1/n) = \sin(\pi k_2/n)$, or $\sin(\pi k_1 n) = -\sin(\pi k_2/n)$. In the first case, $k_1 = k_2$, so $g_{k_1} = g_{k_2}$. The latter cannot occur, as, by assumption, $\sin(\pi k_i/n) > 0$. ∎

Lemma 5.7. *Let F be a face of* $\operatorname{conv} I_2(n)$*. Then F contains at least four linearly independent elements of $I_2(n)$.*

Proof. Since $I_2(n)$ is an irreducible Coxeter group, the origin is in the 4–dimensional interior of $\operatorname{conv} I_2(n)$ by Lemma 3.2. Thus, the elements of $I_2(n)$ on F must compose a hyperplane which does not contain the origin. As a result, F must contain a basis for the entire 4–dimensional space of 2 by 2 matrices. ∎

Lemma 5.8. *Let $l \in \mathbf{Z}$, $0 \le l < n$, with $\sin(2\pi l/n) \neq 0$. Take $c \ge 0$. Suppose $\cos(2\pi k/n) + c\sin(2\pi k/n) \le 1$ for all $k \in \mathbf{Z}$, with equality when $k = l$. Then $l = 1$ and $c = \tan(\pi/n)$.*

Proof. If $c = 0$, then $\cos(2\pi l/n) = 1$, and $\sin(2\pi l/n) = 0$, a contradiction. Therefore $c > 0$. If $\sin(2\pi l/n) < 0$, then $\cos(2\pi(n-l)/n) + c\sin(2\pi(n-l)/n) > \cos(2\pi l/n) + c\sin(2\pi l/n) = 1$, a contradiction. Hence $0 < l < n/2$. Simplifying the given inequality,

$$c \le \frac{1-\cos(2\pi k/n)}{\sin(2\pi k/n)} = \frac{2\sin^2(\pi k/n)}{2\sin(\pi k/n)\cos(\pi k/n)} = \tan\left(\frac{\pi k}{n}\right)$$

for $0 < k < n/2$, with equality if $k = l$. Assume $l \neq 1$. Since $\tan(\pi k/n)$ is increasing on the range $0 < k < n/2$, $c = \tan(\pi l/n) > \tan(\pi/n)$, a contradiction. Therefore, $l = 1$ and $c = \tan(\pi/n)$. ∎

Theorem 5.9. $\operatorname{Extr} I_2(n)^\circ = \mathcal{B}_{I_2(n)}$

Proof. Let $S = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \operatorname{Extr} I_2(n)^\circ$. Left multiplication by elements of $I_2(n)$ preserves rank, and sends $\operatorname{Extr} I_2(n)^\circ$ onto itself by Corollary 5.4. Thus, we may assume that the identity lies on F, the face determined by S. Thus, $\operatorname{Tr}(SI^*) = \operatorname{Tr}(S) = 1$. By Lemma 5.6 and Lemma 5.7, $I_2(n)$ must contain at least 2 reflections. Every reflection of $I_2(n)$ is conjugate to either R_1 or R_2. By Corollary 5.4 we may assume that both the identity and one of R_1 or R_2 lie on F. Now, using conjugation by P as in Corollary 5.5, we may assume that I and R_1 lie on F. Then $(S, I) = w + z = 1$ and $(S, R_1) = w - z = 1$, so $w = 1$, $z = 0$. Using Corollary 5.3 if necessary, assume $y \ge x$. Finally, using Corollary 5.4, if $x < 0$, multiply from the right by R_1, and, if $y < 0$, multiply from the left by R_1 to ensure $x \ge 0$ and $y \ge 0$.

Let $g_k = \operatorname{Refl}(\pi k/n)$, $0 \le k < n$. Since F contains at most 2 rotations by Lemma 5.6, F must contain at least one reflection g_l besides R_1. Since $S \in I_2(n)^\circ$, then $(g_k, S) \le 1$ for every k. Certainly, $(g_l, S) = 1$ because g_l lies on F. Expanding the inner product of g_k with S shows $\cos(2\pi k/n) + (x+y)\sin(2\pi k/n) \le 1$ with equality when $k = l$. By assumption, $x + y \ge 0$. Also, $\sin(2\pi l/n) = 0$ implies $g_l = \pm R_1$. However, $g_l \neq R_1$, and $(-R_1, T) = -1$, so neither case is possible. Lemma 5.8 applies, so $x + y = \tan(\pi/n)$.

There must be one more element on F, and it must be a rotation. Let $h_k = \operatorname{Rot}(2\pi k/n)$, $0 \le k < n, k \in \mathbf{Z}$. Denote the remaining rotation on F by h_l. Since $S \in I_2(n)^\circ$, $(h_k, S) \le 1$ for every k. $(h_l, S) = 1$ because h_l lies on F. Expanding the inner product of h_k with S, obtain $\cos(2\pi k/n) + (y - x)\sin(2\pi k/n) \le 1$, with equality when $k = l$. As before, $\sin(2\pi l/n) = 0$ readily leads to a contradiction, and by assumption, $y - x \ge 0$. Therefore, Lemma 5.8 implies $y - x = \tan(\pi/n)$.

Combining the above information, we see that $x + y = y - x = \tan(\pi/n)$, so $x = 0$ and $y = \tan(\pi/n)$. Therefore, S is a rank 1 operator.

Thus, Theorem 3.4 implies that every element of Extr $I_2(n)^\circ$ is in $\mathcal{B}_{I_2(n)}$. ■

The following corollary is immediate from the proof of the above theorem.

Corollary 5.10. *Every face of* conv $I_2(n)$ *contains exactly two reflections and two rotations in* $I_2(n)$.

6. Extreme elements of $(D_4)^\circ$

Theorem 6.1.

$$\mathrm{Extr}\,(D_4)^\circ = \mathcal{B}_{D_4} \bigcup \{gAh : g, h \in D_4\},$$

where

$$A = \frac{1}{4}\begin{pmatrix} -2 & 2 & 0 & -1 \\ 2 & -2 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a rank 3 *matrix.*

This result was obtained with the help of a computer calculation. We used the cdd program, written by Komei Fukuda. This program is available from
`http://www.ifor.math.ethz.ch/ifor/staff/fukuda/cdd_home/cdd.html`.

After the matrix A is presented, it is not hard to check by hand that it belongs to $(D_4)^\circ$ and explicitly find 16 elements of D_4 whose scalar product with this matrix is exactly 1. These elements are linearly independent and their convex hull is the related face of conv D_4. So, the proof of the Conjecture for the group D_4 does not depend upon the use of computer calculations, though we do not know how to obtain this matrix A without such calculations.

7. Coxeter groups with branching graphs

Theorem 7.1. *Let G be a finite irreducible Coxeter group with a branching Coxeter graph $\Gamma(G)$. Then not all elements of* Extr (conv $G)^\circ$ *are of rank* 1*, i.e.,* $\mathcal{B}_G \subsetneq$ Extr (conv $G)^\circ$.

Proof. It is known from the classification of connected Coxeter graphs (see, e.g., [3]) that every branching Coxeter graph contains a (branching connected) graph $\Gamma(D_4)$ as a subgraph. The statement of Theorem is valid for this group — see the previous Section. So we may assume that $\Gamma(G) \neq \Gamma(D_4)$. Therefore there exists an end vertex π such that the graph $\Gamma(G) \setminus \{\pi\}$ is a branching connected Coxeter graph.

Claim. If all elements of Extr (conv $G)^\circ$ are of rank 1, then the same is true for Extr (conv $H)^\circ$ where H is a Coxeter group such that $\Gamma(H) = \Gamma(G) \setminus \{\pi\}$.

This claim, together with the above considerations, easily leads to a proof of the Theorem.

So, let all elements of Extr (conv $G)^\circ$ be of rank 1. Let ω be an extremal fundamental weight associated with the vertex π. We may assume that its length is 1.

Consider the Coxeter group $\mathrm{Stab}_G\, \omega|_{\omega^\perp}$ and denote it H. Then

$$\Gamma(H) = \Gamma(G) \setminus \{\pi\}$$

Since π is an end vertex, this graph is connected and therefore H is an irreducible group. Consider the hyperplane $\Pi = \{T \in \mathrm{End}\, V : (T, \omega \otimes \omega) = 0\}$. Note that

$$\mathrm{Stab}_G\, \omega = G \cap (\Pi + \mathbf{I})$$

This immediately follows from the fact that operators from G are orthogonal. Also, the affine hyperplane $(\Pi + \mathbf{I})$ is a support hyperplane of the polyhedron $\mathrm{conv}\, G$, i.e.,

$$G \subset \{T \in \mathrm{End}\, V : (T, \omega \otimes \omega) \leq \langle \omega, \omega \rangle\}$$

Therefore the faces of maximal dimension of the polyhedron $\mathrm{conv}(\mathrm{Stab}_G\, \omega)$ are intersections of faces of $\mathrm{conv}\, G$ with the hyperplane $\Pi + \mathbf{I}$. We have assumed that the normals to all faces of $\mathrm{conv}\, G$ are of rank 1. We obtain the group H from the group $\mathrm{Stab}_G\, \omega$ by restricting the action of the latter to its invariant subspace $\omega^\perp$. We can view this as follows:

Let P denote the orthogonal projection of V onto the subspace $\omega^\perp$. Then the operator $T \to PTP$ is an orthogonal projection in End V. Then $H = P(\mathrm{Stab}_G\, \omega)P$. Therefore the faces of maximal dimension of $\mathrm{conv}\, H$ are projections of the faces of maximal dimension of $\mathrm{conv}(\mathrm{Stab}_G\, \omega)$. Therefore the normals to faces of $\mathrm{conv}(H)$ are of the form PbP, where $b \in \mathrm{Extr}\,(\mathrm{conv}\, G)^\circ$. But all these tensors are of rank 1. So the Claim is proven, which completes the proof of Theorem. ■

8. Open Problems

Problem 8.1. *Does the Conjecture 1.1 hold for H_4?*

Problem 8.2. *Find a "classification-free" proof of the Conjecture 1.1.*

Problem 8.3. *Calculate* Extr G° *for irreducible branching Coxeter groups.*

Problem 8.4. *Calculate* Extr env G *for irreducible branching Coxeter groups.*

References

[1] C. Benson and L. Grove, *Finite reflection groups*, Springer-Verlag, New York, 1985.
[2] G. Birkhoff, *Tres observationes sobre el algebra lineal*, University Nac. Tucuman Rev. Ser. **A5** (1946), 147–150.
[3] N. Bourbaki, *Groupes et algebras de Lie, Ch. IV-VI*, Hermann, Paris, 1968.
[4] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.
[5] B. Mityagin, *An interpolation theorem for modular spaces*, Lecture Notes in Mathematics **1070** (1984), 10–23.
[6] N. Zobin and V. Zobina, *Duality in operator spaces and problems of interpolation of operators*, Pitman Research Notes in Math. **25** (1992), 123–144.
[7] ———, *Coxeter groups and interpolation of operators*, Integral Equations and Operator Theory **18** (1994), 335–367.
[8] V. Zobina, *Operator interpolation in spaces with prescribed symmetries*, Ph.D. thesis, Kazan State University, Kazan, Russia, 1979.

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