Finite Reflection Groups and Linear Preserver Problems

Chi-Kwong Li[1], Ilya Spitkovsky[2], and Nahum Zobin
Department of Mathematics, The College of William and Mary,
P.O.Box 8795, Williamsburg, VA 23187-8795
E-mail: ckli@math.wm.edu, ilya@math.wm.edu, zobin@math.wm.edu

Abstract Let G be one of the Coxeter groups $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{D}_n$, or $\mathbf{I}_2(n)$, naturally acting on a Euclidean space V, and let $\mathcal{L}(G)$ stand for the set of linear transformations ϕ of $\operatorname{End} V$ that satisfy $\phi(G) = G$. It is easy to see that $\mathcal{L}(G)$ contains all transformations of the form $X \mapsto PXQ$, $X \mapsto PX^*Q$, where P, Q belong to the normalizer of G in the orthogonal group and $PQ \in G$. We show that in most cases these transformations exhaust $\mathcal{L}(G)$; the only (rather unexpected) exception is the case $G = \mathbf{B}_n$.

1. Introduction

Let G be a finite irreducible Coxeter group naturally acting on a finite dimensional real Euclidean space V; see [2, 4] for related definitions and terminology. The facial structure of the polytope $\operatorname{conv} G$ (the convex hull of G) was recently studied in [5, 13]. In the present paper we address the linear symmetries of $\operatorname{conv} G$ — the linear transformations of the space $\operatorname{End} V$ of linear operators on E preserving the polytope $\operatorname{conv} G$ or, equivalently, preserving G. The problem of describing the set $\mathcal{L}(S)$ of linear transformations of $\operatorname{End} V$ preserving a given set $S \subset \operatorname{End} V$ is an example of linear preserver problems, studied by many researchers, see. e.g., [14].

One can find many simple transformations belonging to $\mathcal{L}(G)$, e.g., left and right multiplications by elements of G, and the operation $T \mapsto T^*$ of taking the adjoint operator obviously belong to $\mathcal{L}(G)$. In fact, the following result can be readily verified for any subgroup G of the orthogonal group $O(V)$:

Lemma 1.1. *Let P, Q belong to the normalizer $N(G)$ of G in the orthogonal group $O(V)$, and assume that $PQ \in G$. Then the transformations $X \mapsto PXQ$ and $X \mapsto PX^*Q$ are in $\mathcal{L}(G)$. These transformations constitute a group.*

[1]Research partially supported by NSF grant DMS 0071994.
[2]Research partially supported by NSF grant DMS 9988579.

The maps $X \mapsto PXQ$ and $X \mapsto PX^*Q$ are called **rigid embeddings**, see [7]. Let $\mathcal{RE}(G)$ denote the group described in Lemma 1.1. Then, according to this Lemma,

$$\mathcal{RE}(G) \subset \mathcal{L}(G).$$

As we shall see, these two sets often coincide: $\mathcal{L}(G) = \mathcal{RE}(G)$; in our study only $G = \mathbf{B}_n$ delivers an unexpected counterexample.

Evidently, if P and Q are invertible operators, then transformations of $\operatorname{End} V$ of the form $X \mapsto PXQ$ or $X \mapsto PX^*Q$ preserve ranks, i.e., $\operatorname{rank}(PXQ) = \operatorname{rank}(PX^*Q) = \operatorname{rank} X$. It is known (see, e.g., [14, Chapter 2]) that these are the only rank-preserving linear transformations of $\operatorname{End} V$. So, $\mathcal{RE}(G)$ may be described as the rank-preserving part of $\mathcal{L}(G)$. Furthermore, if ϕ is **unital**, i.e., ϕ sends the identity operator on $\operatorname{End} V$ to itself, then such ϕ belonging to $\mathcal{RE}(G)$ will be of the form $X \mapsto PXP^{-1}$ or $X \mapsto PX^*P^{-1}$. These transformations are automorphisms or anti-automorphisms of the group G, i.e., they preserve the group structure. Thus, the equality $\mathcal{L}(G) = \mathcal{RE}(G)$ (if it holds) means that the linear transformations preserving G actually preserve much more.

The usual scalar product $(T, S) = \operatorname{tr}(TS^*)$ turns $\operatorname{End} V$ into a Euclidean space. One can show (see Lemma 2.1 below) that every transformation ϕ of the space $\operatorname{End} V$ sending $\operatorname{conv} G$ onto itself has to be orthogonal with respect to this scalar product. So, $\mathcal{L}(G)$ is in fact a subgroup of the group $O(\operatorname{End} V)$ of orthogonal transformations of $\operatorname{End} V$.

The set G is a subset of a Euclidean sphere (of radius $\sqrt{\dim V}$) and thus coincides with the set of the extreme points of its convex hull: $\operatorname{Extr}(\operatorname{conv} G) = G$. This, in turn, implies that $\mathcal{L}(G) = \mathcal{L}(\operatorname{conv} G)$.

For $U \subset \operatorname{End} V$ there is a standard notion of the **polar set**

$$U^\circ = \{T \in \operatorname{End} V \colon (T, S) \leq 1 \ \forall S \in U\}.$$

One can easily see that the set U° is closed and convex, and it contains the origin. Furthermore, $U^\circ = (\operatorname{conv} U)^\circ$. It is well known that studying a convex set U together with its polar set U° is helpful in many problems of convex geometry.

The previous considerations and the orthogonality of transformations from $\mathcal{L}(G)$ imply that the following sets coincide:

$$\mathcal{L}(G) = \mathcal{L}(\operatorname{conv} G) = \mathcal{L}(G^\circ) = \mathcal{L}(\operatorname{Extr}(G^\circ)).$$

Obviously, in our study it would be helpful to know the set $\operatorname{Extr}(G^\circ)$, and we do know this set in many cases, see [5, 13]. To describe it we need some additional definitions and notation.

For a subgroup G of the linear group $GL(V)$ define the **envelope** of G as follows:

$$\operatorname{env} G = \{T \in \operatorname{End} V : TU \subset U \\ \text{for every convex closed } G\text{-invariant subset } U \text{ of } V\}.$$

One readily checks that $\operatorname{env} G$ is a closed convex semigroup of operators, containing $\operatorname{conv} G$. This semigroup naturally arises in the theory of operator interpolation, see [16, 17]. When G is a finite irreducible Coxeter group there exists (see [16]) a convenient dual description of the semigroup $\operatorname{env} G$:

$$\operatorname{Extr}((\operatorname{env} G)^\circ) = \mathcal{B}(G),$$

where the set $\mathcal{B}(G)$ of **Birkhoff tensors**, introduced in [5, 13], is defined as follows:

$$\mathcal{B}(G) = \{\omega \otimes \tau / m_G(\omega, \tau) \colon \omega, \tau \text{ are weights of } G \text{ associated with} \\ \text{distinct end vertices of the Coxeter graph}\}.$$

Here $m_G(x, y) = \max_{g \in G} \langle gx, y \rangle$. See [5, 13] for an explanation of the relations between Birkhoff tensors and the famous Birkhoff Theorem [3] about doubly stochastic matrices.

It is "almost" known that if the Coxeter graph $\Gamma(G)$ of the group G is not branching then in fact $\operatorname{conv} G = \operatorname{env} G$, here "almost" means that the only finite irreducible Coxeter group with a non-branching graph for which this is not yet proven is the group $\mathbf{H}_4$, see [5, 13]. Therefore for every Coxeter group G with a non-branching graph (except for, possibly, $\mathbf{H}_4$) we have

$$\operatorname{Extr}(G^\circ) = \operatorname{Extr}((\operatorname{conv} G)^\circ) = \operatorname{Extr}((\operatorname{env} G)^\circ) = \mathcal{B}(G).$$

So, in this case we have $\mathcal{L}(G) = \mathcal{L}(\mathcal{B}(G))$. Note that the set $\mathcal{B}(G)$ by definition consists only of rank 1 operators, therefore it is usually easier to deal with. As for finite irreducible Coxeter groups with branching graphs, it is known [5, 13] that $\operatorname{conv} G \neq \operatorname{env} G$. The calculation of $\mathcal{L}(\operatorname{env} G)(= \mathcal{L}(\mathcal{B}(G)))$ is then an interesting problem by itself.

All finite irreducible Coxeter groups are classified (see, e.g., [4]), and it is known that there exist four infinite families of Coxeter groups $\mathbf{A}_n, \mathbf{B}_n, \mathbf{D}_n, \mathbf{I}_2(n)$, and six exceptional groups $\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4, \mathbf{H}_3, \mathbf{H}_4$. Groups $\mathbf{A}_n, \mathbf{B}_n, \mathbf{I}_2(n), \mathbf{F}_4, \mathbf{H}_3, \mathbf{H}_4$ have non-branching graphs, the rest $(\mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8)$ have branching graphs. So,

(1) $\quad$ if $G = \mathbf{A}_n, \mathbf{B}_n, \mathbf{I}_2(n), \mathbf{F}_4,$ or $\mathbf{H}_3,$ then $\mathcal{L}(G) = \mathcal{L}(\mathcal{B}(G))$.

In this paper we give a complete description of the set $\mathcal{L}(G)$ for each of the four **infinite families** of finite irreducible Coxeter groups. We have partial results for the six exceptional groups but it seems that some additional techniques is needed for a complete solution of the problem, see Section 7. Only one family of those we study here – the family $\mathbf{D}_n$ – has branching graphs, so $\operatorname{conv}\mathbf{D}_n \neq \operatorname{env}\mathbf{D}_n$. Nonetheless, we show that $\mathcal{L}(\mathbf{D}_n) = \mathcal{L}(\operatorname{env}\mathbf{D}_n) = \mathcal{RE}(\mathbf{D}_n)$ (see Section 5).

Let us mention several known related results. For $G = O(V)$, Wei [15] showed that $\mathcal{L}(O(V)) = \mathcal{RE}(O(V))$, i.e.,

$$\phi \in \mathcal{L}(O(V)) \quad \text{if and only if} \quad \phi(A) = PAQ \quad \text{or} \quad \phi(A) = PA^*Q$$

for some orthogonal operators P and Q. The case $G = \mathbf{A}_n$ was studied in [11], and it was shown that $\mathcal{L}(\mathbf{A}_n) = \mathcal{RE}(\mathbf{A}_n)$. We give a different proof of this result below.

Note that one may also study linear transformations ϕ of $\operatorname{End} V$ such that $\phi(G) \subset G$, not necessarily $\phi(G) = G$. When $G = O(V)$, one gets the same conclusion as above except when $\dim V = 2, 4, 8$, and there exist singular transformations in these cases, see [15] for details.

For the preserver results for other classical linear groups, see [14, Section 4.6].

Our interest in linear symmetries of the $\mathcal{RE}(G)$-invariant convex polytopes $\operatorname{conv} G$ (and $\operatorname{env} G$) was mostly motivated by a desire to understand the geometry of general $\mathcal{RE}(G)$-invariant convex bodies — the unit balls of $\mathcal{RE}(G)$-invariant norms. If we consider a complex Hilbert space V and take G to be the group $U(V)$ of unitary operators, then $\mathcal{RE}(U(V))$-invariant norms are called **unitarily invariant norms** (see, e.g., [9]). These norms are closely connected with the Schatten – von Neumann ideals, and they have been studied by many authors, see [9]. Since Euclidean balls in V are obviously the only $U(V)$-invariant convex closed sets in V, then $\operatorname{env} U(V)$ is simply the unit ball of the operator norm, which in turn coincides with $\operatorname{conv} U(V)$. So,

$$\operatorname{Extr}(\operatorname{env} U(V))^\circ = \{x \otimes y \in \operatorname{End} V : \langle x, x\rangle\langle y, y\rangle = 1\} = \mathcal{B}(U(V)).$$

Therefore, $\operatorname{conv}\mathcal{B}(U(V))$ is the unit ball of the norm dual to the operator norm, which is the nuclear (or trace) norm. The operator norm and the nuclear norm are very important examples of unitarily invariant norms. In particular, it is known that every unitarily invariant norm is an interpolation norm for this couple, see, e.g., [9]. These two norms are natural "non-commutative" analogs of $\mathbf{B}_n$-invariant norms l_∞ and l_1. It was shown in [16] that if G is an irreducible Coxeter group whose graph is non-branching and if ω_i, $i = 1, 2$, are the weights of G associated with distinct end vertices of the Coxeter graph, then $\operatorname{conv}\operatorname{Orb}_G \omega_i$ are the unit balls of G-invariant (pseudo)norms analogous to the l_∞-

and l_1-norms. So, the sets $\operatorname{conv} G$ and $\operatorname{conv} \mathcal{B}(G)$ whose linear symmetries we study in this paper can be viewed as the unit balls of the "non-commutative" (or "quantum") versions of these (pseudo)norms.

Isometries of $\mathcal{RE}(U(V))$- and $\mathbf{B}_n$-invariant norms have been studied by many authors, see, e.g., [8, 12, 10].

It is important to explore the possible isometries of general $\mathcal{RE}(G)$-invariant norms. In particular, it would be very interesting to compute the isometries of the $\mathcal{RE}(G)$-invariant norms whose unit balls are the following convex bodies: $\operatorname{conv} \operatorname{Orb}_{\mathcal{RE}(G)} C$ for $C \in \operatorname{End} V$. Note that if $C \in G$, then $\operatorname{Orb}_{\mathcal{RE}(G)} C = G$, and if $C \in \mathcal{B}(G)$ then (at least in the case of a non-branching graph) $\operatorname{Orb}_{\mathcal{RE}(G)} C = \mathcal{B}(G)$. Therefore the problems we address in this paper can be described as exploring the isometries of the basic $\mathcal{RE}(G)$-invariant norms, namely those whose unit balls are $\operatorname{conv} \operatorname{Orb}_{\mathcal{RE}(G)} C$ where $C \in G$ or $C \in \mathcal{B}(G)$, for the infinite families of irreducible Coxeter groups. We anticipate that it would be very difficult to solve such a problem for a general $C \in \operatorname{End} V$.

2. Preliminary Results

Recall that for a set $H \subset \operatorname{End} V$ its **commutant** is defined as the set of all operators on V commuting with each operator from H. The commutant is said to be **trivial** if it consists only of scalar operators. It is well known that the commutant of a subgroup of the unitary group in a complex finite dimensional Hilbert space is trivial if and only if this group is irreducible, or, equivalently, the linear span of this subgroup is the whole $\operatorname{End} V$. The situation of a subgroup of the orthogonal group in a real finite dimensional Euclidean space is more complicated: if the linear span of the subgroup is the whole $\operatorname{End} V$ then certainly the commutant is trivial and the group is irreducible, but not vice versa — an irreducible subgroup may span a proper subspace of $\operatorname{End} V$, and its commutant may be non-trivial. A good example is delivered by the group of rotations by multiples of $\pi/4$ in $\mathbb{R}^2$. Nevertheless, it is known (see, e.g., [5, 13]) that an irreducible Coxeter group spans the whole $\operatorname{End} V$, so its commutant is trivial.

Evidently, for every closed subset U of V the set $\mathcal{L}(U)$ is a closed semigroup. Often one can prove much more:

Lemma 2.1. *Let U be a subset of V spanning the whole V. Then the set $\mathcal{L}(U)$ is actually a group. Assume, in addition, that U is compact and that $\mathcal{L}(U)$ contains a subgroup of orthogonal operators whose commutant is trivial. Then $\mathcal{L}(U)$ itself is a closed subgroup of the orthogonal group $O(V)$.*

Proof. The fact that operators from $\mathcal{L}(U)$ are invertible immediately follows from the condition that U spans the whole space V. Therefore $\mathcal{L}(U)$ is a subgroup of the linear group $GL(V)$, and it is obviously closed.

If U is bounded then so is $\mathcal{L}(U)$. Therefore there exists a positive definite operator $T \in GL(V)$ such that $T(\mathcal{L}(U))T^{-1}$ is a subgroup of $O(V)$ (see, e.g., [1, 6]). Hence, for any $\phi \in \mathcal{L}(U) \cap O(V)$, we see that $T\phi T^{-1}(T\phi T^{-1})^*$ is the identity operator, and hence $T^2\phi = \phi T^2$. Since the commutant of $\mathcal{L}(U) \cap O(V)$ is assumed to be trivial then T^2 is a scalar operator. Therefore T, which is the positive square root of T^2, is also a scalar operator. Hence, $\mathcal{L}(U) = T(\mathcal{L}(U))T^{-1}$ is a subgroup of $O(V)$. ∎

Corollary 2.2. *Let G be a subgroup of the orthogonal group $O(V)$, spanning the whole space* $\operatorname{End} V$. *Then* $\mathcal{L}(G) \subset O(\operatorname{End} V)$.

Proof. By Lemma 2.1, we only need to present an orthogonal subgroup of $\mathcal{L}(G)$ with a trivial commutant. Consider the group $G \times G$, acting on $\operatorname{End} V$ by left and right multiplications. Obviously, it is a part of $\mathcal{L}(G)$. Since G spans the $(\dim V)^2$-dimensional space $\operatorname{End} V$ then one can show that $G \times G$ spans a $(\dim V)^4$-dimensional space, i.e., the whole $\operatorname{End}(\operatorname{End} V)$. This excludes the possibility of a nontrivial commutant. ∎

Corollary 2.3. *Let G be a finite irreducible Coxeter group, naturally acting on V. Then* $\mathcal{L}(\mathcal{B}(G)) \subset O(\operatorname{End} V)$.

Proof. Since $\mathcal{B}(G)$ spans $\operatorname{End} V$ (see [5, 13]), it suffices to present an orthogonal subgroup of $\mathcal{L}(\mathcal{B}(G))$ with a trivial commutant. We may again choose the group $G \times G$. ∎

Lemma 2.4. *The group $\mathcal{RE}(G)$ acts on the set $\mathcal{B}(G)$.*

Proof. According to [16],

$$\mathcal{B}(G) = \operatorname{Extr}(\operatorname{conv}(G^\circ \bigcap \{ \text{ rank 1 tensors } \})).$$

Thus, it suffices to show that $\mathcal{RE}(G)$ maps $G^\circ \cap \{ \text{ rank 1 tensors } \}$ into itself. To this end, consider $x \otimes y \in G^\circ$ (i.e., such that $\forall\, g \in G,\ (g, x \otimes y) \leq 1$). Then for all $P, Q \in N(G)$ satisfying $PQ \in G$ and for any $g \in G$:

$$(g, P(x \otimes y)Q) = (P^* g Q^*, x \otimes y) = ((P^{-1}gP)(QP)^{-1}, x \otimes y) \leq 1,$$

because $P^{-1}gP \in G$ and $QP = Q(PQ)Q^{-1} \in G$. ∎

Strategy of Proofs

Let us outline the approach we are going to use to calculate the groups $\mathcal{L}(G)$ and $\mathcal{L}(\operatorname{env} G) = \mathcal{L}(\mathcal{B}(G))$.

Let U be a finite subset of $\operatorname{End} V$, spanning the whole space $\operatorname{End} V$. Assume that the group $\mathcal{RE}(G)$ acts on U, i.e., for every $T \in \mathcal{RE}(G)$ and every $u \in U$ we have $Tu \in U$. This assumption obviously holds for $U = G$, and, due to Lemma 2.4, this assumption is also true for $U = \mathcal{B}(G)$. Moreover, this action is transitive in all cases we study here — this is obvious for $U = G$, also obvious for $U = \mathcal{B}(G)$, provided $\Gamma(G)$ is non-branching; not so obvious for $U = \mathcal{B}(\mathbf{D}_n)$, see Lemma 5.4.

Take any $\phi \in \mathcal{L}(U)$. Choose $u_0 \in U$ (if $U = G$ then it is natural to choose $u_0 = I$). If $\phi(u_0) \notin \mathcal{RE}(G)u_0$ then definitely $\mathcal{RE}(G) \neq \mathcal{L}(U)$. Certainly, this cannot happen in the cases we study in this paper since in all these cases the action of $\mathcal{RE}(G)$ is transitive on U, but we cannot exclude such a possibility for $G = \mathbf{E}_k,\ k = 6, 7, 8$.

If $\phi(u_0) \in \mathcal{RE}(G)u_0$, then let $T \in \mathcal{RE}(G)$ be such that $T\phi(u_0) = u_0$. Then $\phi_1 = T\phi \in \mathcal{L}(U)$, and ϕ_1 fixes u_0.

Let

$$U_1 = \{u \in U : (u, u_0) = a_1 \neq (u_0, u_0)\}.$$

We usually choose $a_1 = \max_{v \neq u_0}(v, u_0)$. Since ϕ_1 preserves the scalar product and fixes u_0, then $\phi_1(U_1) = U_1$. The subgroup

$$\operatorname{Stab} u_0 = \{T \in \mathcal{RE}(G) : Tu_0 = u_0\}$$

also acts on U_1. Choose $u_1 \in U_1$ and consider $\phi_1(u_1) \in U_1$. If $\phi_1(u_1) \notin (\operatorname{Stab} u_0)u_1$, then $\mathcal{L}(U) \neq \mathcal{RE}(G)$. Otherwise, take $T_1 \in \operatorname{Stab} u_0$ such that $T_1\phi_1(u_1) = u_1$. Then $\phi_2 = T_1\phi_1 \in \mathcal{L}(U)$ fixes both u_0 and u_1.

Continuing this procedure we either find out that $\mathcal{L}(U) \neq \mathcal{RE}(G)$, or deduce that $\phi_k \in \mathcal{L}(U)$ fixes so many elements that it is only possible if $\phi_k = I$. In the latter case, $\phi \in \mathcal{RE}(G)$.

In the body of the proofs, we will repeatedly make use of characterizations of various subsets of U in terms of scalar products. We will mark those characterizations as "claims". Once stated, each such claim can be justified by a straightforward (though sometimes lengthy) computation.

Our investigation of the sets $\mathcal{L}(G)$ for Coxeter groups G is a case by case study, in which we are using explicit matrix realizations for the groups G and explicit formulas for their simple roots and fundamental weights in special orthonormal bases given in [2, 4]. We use these formulas in the sections below without further references. Our space V is $\mathbb{R}^n$ (or a hypersubspace of $\mathbb{R}^{n+1}$ — in the case $G = \mathbf{A}_n$). In what follows, we abbreviate $O(\mathbb{R}^n)$ to $O(n)$, denote by $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$, and let $e = \sum_{j=1}^n e_j$. All operators will be described by their matrices in the standard basis. In particular, for $E_{ij} = e_i \otimes e_j$, its matrix in the standard basis is $e_i e_j^t$. As usual, T^t denotes the transpose of the matrix T. We let $\mathbf{M}_n(\mathbb{R})$ denote the space of real $n \times n$ matrices.

3. Groups $\mathbf{A}_n$

Matrix Realization

For $n \geq 2$, let Perm_{n+1} be the group of $(n+1) \times (n+1)$ permutation matrices. Consider the following subspace V of $\mathbb{R}^{n+1}$:

$$V = \{v = (v_1, \dots, v_{n+1})^t \in \mathbb{R}^{n+1} : v_1 + \cdots + v_{n+1} = 0\}.$$

This subspace is invariant under the action of the group Perm_{n+1}. Every matrix $P \in \mathrm{Perm}_{n+1}$ can be rewritten in the form

$$P = (P - ee^t/(n+1)) + ee^t/(n+1) = P_V + P_{V^\perp}.$$

Obviously, $P_V e = 0, e^t P_V = 0$, and $P_V v = Pv$ for every $v \in V$. Group $\mathbf{A}_n$ is the group of restrictions to V of operators from Perm_{n+1}. So, operators $P|_V$ from $\mathbf{A}_n$ can be identified with $(n+1) \times (n+1)$ matrices P_V, where $P \in \mathrm{Perm}_{n+1}$. Note that matrices P_V have zero row and column sums. Therefore the space $\mathrm{End}\, V$ (which coincides with the linear span of $\mathbf{A}_n$) is naturally identified with the space $\mathbf{M}_n^0$ of $(n+1) \times (n+1)$ real matrices with zero row and column sums. The natural scalar product on $\mathrm{End}\, V$ is given by the usual formula $(X, Y) = \mathrm{tr}(XY^t)$.

Birkhoff Tensors

Let $w = (n+1)e_1 - e = (n, -1, \dots, -1)^t \in V$. Then

$$(n+1)\mathcal{B}(\mathbf{A}_n) = \{-Pww^tQ : P, Q \in \mathbf{A}_n\} \subset \mathbf{M}_n^0.$$

Linear Preservers

Li, Tam and Tsing [11] showed that a linear preserver of $\mathbf{A}_n$ on $\mathbf{M}_n^0$ must be of the form $A \mapsto UAV$ or $A \mapsto UA^tV$ for some $U, V \in \mathbf{A}_n$. Here we give a different proof.

Theorem 3.1. *Let $n \geq 2$. Then*

$$\mathcal{L}(\mathbf{A}_n) = \mathcal{L}(\mathcal{B}(\mathbf{A}_n)) = \mathcal{RE}(\mathbf{A}_n).$$

In other words, the following statements are equivalent for a linear transformation ϕ of $\mathbf{M}_n^0$:

(a) $\phi(\mathbf{A}_n) = \mathbf{A}_n$.
(b) $\phi(\mathcal{B}(\mathbf{A}_n)) = \mathcal{B}(\mathbf{A}_n)$.
(c) *There exist $U, V \in \mathbf{A}_n$ such that ϕ is of the form*

$$A \mapsto UAV \qquad \text{or} \qquad A \mapsto UA^tV.$$

Proof. By (1), (a) and (b) are equivalent. Clearly, (c) implies (b). It remains to prove that (b) implies (c). So, let $\phi(\mathcal{B}(\mathbf{A}_n)) = \mathcal{B}(\mathbf{A}_n)$.

Let $\mathbf{S} = -(n+1)\mathcal{B}(\mathbf{A}_n) = \{Pw(Qw)^t : P, Q \in \mathbf{A}_n\}$. Then $\phi(\mathbf{S}) = \mathbf{S}$. We may assume that $\phi(ww^t) = ww^t$ (otherwise, replace ϕ by a transformation of the form $X \mapsto P\phi(X)Q$ for some suitable $P, Q \in \mathbf{A}_n$).

Now for $j = 1, \dots, n+1$, consider $v_j = (n+1)e_j - e = P_j w,\ P_j \in \mathbf{A}_n$. Then, obviously, $w = v_1$, and

$$\mathbf{S} = \{A_{ij} = v_i v_j^t : 1 \leq i, j \leq n+1\}.$$

Note that $v_p^t v_p = n^2 + n$ and $v_p^t v_q = -(n+1)$, if $p \neq q$, therefore

$$(A_{ij}, A_{ks}) = -(n^2+n)(n+1) \text{ if and only if } i = k, j \neq s, \text{ or } i \neq k, j = s.$$

Since ϕ preserves the inner product, we get

$$(A_{11}, \phi(A_{12})) = (\phi(A_{11}), \phi(A_{12})) = (A_{11}, A_{12}) = -(n^2+n)(n+1),$$

so we conclude that

(i) $\phi(A_{12}) = A_{1j}$,

or

(ii) $\phi(A_{12}) = A_{j1} = A_{1j}^t$

for some $j \geq 2$.

We may assume that (i) holds; otherwise, replace ϕ by a transformation of the form $X \mapsto \phi(X)^t$ (this will not destroy the condition $\phi(A_{11}) = A_{11}$ since $A_{11}^t = A_{11}$). Furthermore, we may assume that $\phi(A_{12}) = A_{12}$; otherwise, replace ϕ by a transformation of the form $X \mapsto \phi(X)Q$ where $Q \in \mathbf{A}_n$ corresponds to the transposition interchanging 2 and j (again, the condition $\phi(A_{11}) = A_{11}$ is preserved). Now, $\phi(A_{11}) = A_{11}$ and $\phi(A_{12}) = A_{12}$. Let $j = 1, 2$, then

$$(A_{1j}, \phi(A_{13})) = (\phi(A_{1j}), \phi(A_{13})) = (A_{1j}, A_{13}) = -(n^2+n)(n+1),$$

and hence $\phi(A_{13}) = A_{1s}$ for some $s \geq 3$. Again, we may assume that $\phi(A_{13}) = A_{13}$. Repeating these arguments, we modify ϕ until we get $\phi(A_{1j}) = A_{1j}$ for all $j = 1, \dots, n+1$.

Since $\phi(A_{1j}) = A_{1j}$ for all $j = 1, \dots, n+1$, we see that for any $s \geq 2$

$$\phi(A_{s1}) \notin \{A_{1k} : 1 \leq k \leq n+1\},$$

and

$$(\phi(A_{s1}), A_{11}) = (\phi(A_{s1}), \phi(A_{11})) = (A_{s1}, A_{11}) = -(n^2+n)(n+1).$$

Therefore for any $s \geq 2$ we have $\phi(A_{s1}) = A_{\sigma(s),1}$ for some permutation σ of $\{1, 2, \cdots, n+1\}$, $\sigma(1) = 1$. We may assume that $\phi(A_{s1}) = A_{s1}$; otherwise, replace ϕ by a mapping of the form $X \mapsto Q\phi(X)$ where

$Q \in \mathbf{A}_n$ corresponds to the permutation σ^{-1} (this will not destroy the equalities $\phi(A_{1j}) = A_{1j}$, $1 \le j \le n+1$, since $\sigma(1) = 1$).

Since ϕ fixes A_{1j}, A_{j1}, $1 \le j \le n+1$, then ϕ maps the set $\{A_{ij} : 2 \le i, j \le n+1\}$ onto itself. Take $A_{ij}, i, j \ge 2$. Then $(\phi(A_{ij}), A_{i1}) = (\phi(A_{ij}), \phi(A_{i1})) = (A_{ij}, A_{i1}) = -(n^2+n)(n+1)$, so $\phi(A_{ij}) = A_{is}, s \ge 2$. Similarly, we obtain that $\phi(A_{ij}) = A_{rj}, r \ge 2$. So, $\phi(A_{ij}) = A_{ij}$. Therefore ϕ fixes all elements of $\mathcal{B}(\mathbf{A}_n)$, i.e., ϕ is the identity. ∎

4. Groups $\mathbf{B}_n$

Matrix Realization

Group $\mathbf{B}_n$ consists of all the $2^n n!$ signed permutation matrices in $\mathbf{M}_n(\mathbb{R})$.

Birkhoff Tensors

The set $\mathcal{B}(\mathbf{B}_n)$ is the set of all matrices of the form $e_j e^t Q$ or Qee_j^t with $Q \in \mathbf{B}_n$.

Linear Preservers

Let $X \circ Y$ denote the Schur (entrywise) product of matrices $X, Y \in \mathbf{M}_n(\mathbb{R})$.

Theorem 4.1. *Let $n > 2$. The following conditions are equivalent for a linear transformation ϕ of $\mathbf{M}_n(\mathbb{R})$.*

(a) $\phi(\mathbf{B}_n) = \mathbf{B}_n$.
(b) $\phi(\mathcal{B}(\mathbf{B}_n)) = \mathcal{B}(\mathbf{B}_n)$.
(c) *There exist $P, Q \in \mathbf{B}_n$ and a matrix S with entries ± 1 such that ϕ is of the form*

$$A \mapsto S \circ (PAQ) \quad \text{or} \quad A \mapsto S \circ (PA^tQ). \tag{2}$$

Proof. By (1), (a) and (b) are equivalent. One readily checks that (c) implies (a) and (b).

Suppose (b) holds. We may assume that

$$\phi(ee_1^t) = ee_1^t, \tag{3}$$

Each matrix X from $\mathcal{B}(\mathbf{B}_n)$ has either exactly one row or exactly one column consisting of ± 1's, with all other entries equal to zero, let us call this row (or column) the **special line** of X. If the special lines of $X_1, X_2 \in \mathcal{B}(\mathbf{B}_n)$ are parallel and non-coinciding then $(X_1, X_2) = 0$. If the special lines of $X_1, X_2 \in \mathcal{B}(\mathbf{B}_n)$ are non-parallel, then $(X_1, X_2) = \pm 1$. So, if for $X_1, X_2 \in \mathcal{B}(\mathbf{B}_n)$ we have (X_1, X_2) different from $0, \pm 1$, then the special lines of X_1, X_2 must coincide. Note that for odd n if $X_1, X_2 \in \mathcal{B}(\mathbf{B}_n)$ have the same special lines, and the special line of X_1 consists of 1's, then $(X_1, X_2) \ne 0$, so all matrices from $X_1^\perp \bigcap \mathcal{B}(\mathbf{B}_n)$ have their special lines parallel to the special line of X_1 but not coinciding with it.

Denote by A_{kj} the matrix obtained from ee_j^t by changing the sign of its (k,j)-entry: $A_{kj} = ee_j^t - 2e_k e_j^t$. Note that A_{kj} and A_{kj}^t are in $\mathcal{B}(\mathbf{B}_n)$.

Claim 1. *Let $n > 3$. Then*

$$\{A_{1j} : 1 \leq j \leq n\} = \{X \in \mathcal{B}(\mathbf{B}_n) : (X, ee_1^t) = n-2\}.$$

Claim 2. *Let $n = 3$. Then*

$$\{A_{1j} : 1 \leq j \leq 3\} = \{X \in \mathcal{B}(\mathbf{B}_3) : (X, ee_1^t) = 1, \text{ and } (X,Y) = 0$$
$$\text{for all } Y \in \mathcal{B}(\mathbf{B}_3) \text{ such that } (Y, ee_1^t) = 0\}.$$

Since ϕ fixes ee_1^t and preserves the inner product on $\mathbf{M}_n(\mathbb{R})$, it follows from Claims 1,2 that ϕ maps the set $\{A_{k_1} : 1 \leq k \leq n\}$ onto itself:

(4) $\quad \phi(A_{k1}) = A_{\sigma(k),1}$ for a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$

Combining (3) with (4) shows that

$$\phi(e_k e_1^t) = e_{\sigma(k)} e_1^t, \ 1 \leq k \leq n.$$

Replacing ϕ by $P\phi$ with an appropriate permutation matrix P we ensure that

$$\phi(e_k e_1^t) = e_k e_1^t, \ 1 \leq k \leq n. \tag{5}$$

Consider $\phi(e_j e^t) \in \mathcal{B}(\mathbf{B}_n)$. Then

$$(\phi(e_j e^t), e_k e_1^t) = (\phi(e_j e^t), \phi(e_k e_1^t)) = (e_j e^t, e_k e_1^t) = \delta_{jk},$$

so the special line of $\phi(e_j e^t)$ contains the $(j,1)$ position of the matrix. Therefore this line is either the j-th row or the first column. The latter cannot be true since $(\phi(e_j e^t), e_k e_1^t) = 0$ for $k \neq j$. So, the special line of $\phi(e_j e^t)$ is the j-th row. Replace ϕ with $S \circ \phi$ with an appropriate matrix S with ± 1 entries, having entries 1 in the first column, — this will not affect the equalities (5) — and we may assume that

$$\phi(e_j e^t) = e_j e^t, \ 1 \leq j \leq n. \tag{6}$$

Replacing $e_j e^t$ in the previous considerations by A_{j1}^t, $j \geq 2$, we conclude that the special lines of $\phi(A_{j1}^t)$, $j \geq 2$, all coincide with the first row. Since $(\phi(A_{j1}^t), e_1 e^t) = (\phi(A_{j1}^t), \phi(e_1 e^t)) = (A_{j1}^t, e_1 e^t) = n-2$, then $\phi(A_{j1}^t) = A_{\sigma(j),1}^t$, where σ is a permutation such that $\sigma(1) = 1$. Replacing ϕ with ϕP, where P is an appropriate permutation matrix fixing e_1, we may assume that $\phi(A_{j1}^t) = A_{j1}^t$, $j \geq 2$. Combining this with (6), (5) we conclude that

$$\phi(e_k e_j^t) = e_k e_j^t, \text{ provided } k = 1 \text{ or } j = 1. \tag{7}$$

Now take any A_{kj}, $k, j \geq 2$. Then $(\phi(A_{kj}), e_1 e_s^t) = (\phi(A_{kj}), \phi(e_1 e_s^t)) = (A_{kj}, e_1 e_s^t) = \delta_{js}$, and $(\phi(A_{kj}), e_s e^t) = (\phi(A_{kj}), \phi(e_s e^t)) = (A_{kj}, e_s e^t) = (-1)^{\delta_{ks}}$. These equalities imply that the j-th column is the special line of $\phi(A_{kj})$, and, moreover, that $\phi(A_{kj}) = A_{kj}$ for $k, j \geq 2$. Together with (7) this means that ϕ is the identity mapping. ■

Remark 1

There are other preservers of $\mathbf{B}_2$, namely, any orthogonal transformation on $\mathbf{M}_2(\mathbb{R})$ mapping the set $\{E_{11} \pm E_{22}, E_{12} \pm E_{21}\}$ into $\mathbf{B}_2$ is admissible. Such a collection of transformations will form a group isomorphic to $\mathbf{B}_4$. Note that group $\mathbf{B}_2$ coincides with $\mathbf{I}_2(4)$, a particular case of the groups $\mathbf{I}_2(n)$ considered in Section 6

Remark 2

For $n > 2$, group $\mathcal{L}(\mathbf{B}_n)$ differs from $\mathcal{RE}(\mathbf{B}_n)$. Actually, one can show that $\mathcal{L}(\mathbf{B}_n)$ is not contained even in the normalizer of $\mathcal{RE}(\mathbf{B}_n)$ as follows. Consider the transformation ψ in $\mathcal{L}(\mathbf{B}_n)$ so that $\psi(A)$ is obtained from A by multiplying its $(1,1)$ entry by -1. Let P be the permutation matrix obtained from I_n by interchanging the first two rows, and let $\phi_P(A) = PA$ for all $A \in \mathbf{M}_n(\mathbb{R})$. Then $\phi_P \in \mathcal{RE}(\mathbf{B}_n)$. Observe, however, that the matrix E of all ones has rank one, while $(\psi^{-1}\phi_P\psi)(E)$ has rank 2 (if $n \geq 3$). Thus, the mapping $\psi^{-1}\phi_P\psi$ does not preserve ranks, and therefore cannot lie in $\mathcal{RE}(\mathbf{B}_n)$. Consequently, ψ is not an element of the normalizer of $\mathcal{RE}(\mathbf{B}_n)$.

5. Groups $\mathbf{D}_n$

Matrix Realization

The group $\mathbf{D}_n$ consists of $2^{n-1}n!$ signed permutation matrices in $\mathbf{M}_n(\mathbb{R})$ with **even** numbers of -1's. One can easily prove that $\mathbf{D}_2$ and $\mathbf{D}_3$ coincide, respectively, with $S_2 \times S_2$ (where S_2 is the two-element group) and $\mathbf{A}_3$. So, it is a standard convention (which we follow) to consider $\mathbf{D}_n$ only for $n \geq 4$.

Possible Inner Products

Let $X \in \mathbf{D}_n$, $X \neq I$. Then $(I, X) \in \{0, \pm 1, \cdots, \pm(n-2)\}$. The equality $(I, X) = n - 2$ holds if and only if $X = I - (e_i \pm e_j)(e_i \pm e_j)^t$ for some $1 \leq i < j \leq n$.

Birkhoff Tensors

Let $w_1 = (n-2)e_1 = (n-2, 0, \dots, 0)^t, w_2 = e = (1, \dots, 1)^t, w_3 = e - 2e_n = (1, \dots, 1, -1)^t$. Let $\mathbf{S}_1$ be the collection of rank one matrices of the form $Pw_1w_2^tQ$ with $P, Q \in \mathbf{D}_n$, $\mathbf{S}_2$ be the collection of rank one matrices of the form $Pw_1w_3^tQ$ with $P, Q \in \mathbf{D}_n$, and $\mathbf{S}_3$ be the collection of rank one matrices of the form $Pw_2w_3^tQ$ with $P, Q \in \mathbf{D}_n$. Furthermore, let

$$\mathbf{S}_i^t = \{A^t : A \in \mathbf{S}_i\} \text{ for } i = 1, 2, 3.$$

Then we have

$$(n-2)\mathcal{B}(\mathbf{D}_n) = \mathbf{S}_1 \cup \mathbf{S}_1^t \cup \mathbf{S}_2 \cup \mathbf{S}_2^t \cup \mathbf{S}_3 \cup \mathbf{S}_3^t.$$

Normalizers

To facilitate our study of $\mathbf{D}_n$, we need a description of its normalizer in $O(n)$. First, recall that (see [2, 8]) $\mathbf{F}_4$ is generated by the group $\mathbf{B}_4$ and the reflection

$$R = 1/2 \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = I - (1/2)ee^t. \tag{8}$$

The normalizer of $\mathbf{F}_4$ in $O(4)$ is the group $N(\mathbf{F}_4)$ generated by $\mathbf{B}_4$ and the operator

$$B = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right). \tag{9}$$

To see that this is indeed the case, note that the normalizer of $\mathbf{F}_4$ in $O(4)$ is a compact overgroup of $\mathbf{F}_4$. By [8], the only compact overgroups of $\mathbf{F}_4$ are $N(\mathbf{F}_4)$ and $O(4)$. One can easily verify that B belongs to the normalizer of $\mathbf{F}_4$, so $N(\mathbf{F}_4)$ is in the normalizer of $\mathbf{F}_4$, and that $O(4)$ is not the normalizer of $\mathbf{F}_4$.

Lemma 5.1. *If $n > 4$, the normalizer $N(\mathbf{D}_n)$ of $\mathbf{D}_n$ in $O(n)$ is $\mathbf{B}_n$. The normalizer $N(\mathbf{D}_4)$ of $\mathbf{D}_4$ in $O(4)$ is $\mathbf{F}_4$.*

Proof. Clearly, the normalizer $N(\mathbf{D}_n)$ of $\mathbf{D}_n$ in $O(n)$ is a compact group. One easily checks that $N(\mathbf{D}_n)$ contains $\mathbf{B}_n$. By the results in [8], if $n \neq 4$, then the only compact overgroup of $\mathbf{B}_n$ is $O(n)$. Clearly, $O(n)$ is not the normalizer. So, $N(\mathbf{D}_n) = \mathbf{B}_n$.

If $n = 4$, then one can easily check that $\mathbf{F}_4 \subset N(\mathbf{D}_4)$ (one only needs to check that $R \in N(\mathbf{D}_4)$) but $N(\mathbf{F}_4) \not\subseteq N(\mathbf{D}_4)$ since $B \notin N(\mathbf{D}_4)$. It is known [8] that the only compact overgroups of $\mathbf{F}_4$ are $N(\mathbf{F}_4)$ and $O(4)$. Thus, $N(\mathbf{D}_4) = \mathbf{F}_4$. ∎

Let P, Q belong to the normalizer $N(G)$ of G in the orthogonal group $O(V)$, and $PQ \in G$, so $X \mapsto PXQ$ and $X \mapsto PX^*Q$ are transformations from $\mathcal{RE}(G)$. Since G is a normal subgroup in $N(G)$ then we may consider the factor group $N(G)/G$. Let $\widetilde{X}$ denote the coset of $X \in G$ in $N(G)/G$. Then the condition $PQ \in G$ can be rewritten as $\widetilde{P}\widetilde{Q} = \widetilde{I}$, i.e., $\widetilde{P}^{-1} = \widetilde{Q}$.

Let us find out what pairs $P, Q \in N(\mathbf{D}_n)$ satisfy $PQ \in \mathbf{D}_n$.

Lemma 5.2. *Let $P, Q \in \mathbf{B}_n$, $n > 4$. Then $PQ \in \mathbf{D}_n$ if and only if $P, Q \in \mathbf{D}_n$ or $P, Q \in \mathbf{B}_n \setminus \mathbf{D}_n$.*

Proof. The group $\mathbf{B}_n$ is generated by $\mathbf{D}_n$ and one reflection $T = I - 2e_n \otimes e_n$, $T^2 = I$, $T \in N(\mathbf{D}_n)$. Then each operator $W \in \mathbf{B}_n \setminus \mathbf{D}_n$ can be written as $W = Tg = hT$, where $g, h \in \mathbf{D}_n$. Therefore for any $P, Q \in \mathbf{B}_n \setminus \mathbf{D}_n$ we always have $PQ \in \mathbf{D}_n$. If only one of P, Q belongs to $\mathbf{B}_n \setminus \mathbf{D}_n$ then obviously $PQ \notin \mathbf{D}_n$. If both P and Q belong to $\mathbf{D}_n$ then obviously $PQ \in \mathbf{D}_n$. ■

According to the definition, the group $\mathbf{F}_4$ is generated by $\mathbf{B}_4$ and the reflection $R = I - (1/2)e \otimes e$ given by (8), or, in other terms, by $\mathbf{D}_4$ and two reflections: $T = I - 2e_4 \otimes e_4$ and R. Therefore, the factor group $\mathbf{F}_4/\mathbf{D}_4$ is generated by two elements — $\widetilde{T}$ and $\widetilde{R}$. Let us calculate the group $\mathbf{F}_4/\mathbf{D}_4$. Let H denote the subgroup in $\mathbf{F}_4$ generated by the two reflections T, R. Since the angle between e_4 and e is $\pi/3$, then H is naturally isomorphic to A_2, so it consists of 6 operators: $I, R, T, RT, TR, RTR (= TRT)$, four of which (I, R, T, RTR) are involutions, and the other two (RT, TR) are not. Since $T \in \mathbf{B}_4 \setminus \mathbf{D}_4$ and $R \in \mathbf{F}_4 \setminus \mathbf{B}_4$ then $H \cap \mathbf{D}_4 = \{I\}$, $H \cap \mathbf{B}_4 = \{I, T\}$. Therefore $\mathbf{F}_4/\mathbf{D}_4$ is naturally isomorphic to H. These considerations lead to the following result:

Lemma 5.3. *Let P, Q belong to $\mathbf{F}_4$. Then $PQ \in \mathbf{D}_4$ i.e., $\widetilde{P}\widetilde{Q} = \widetilde{I}$, if and only if one of the following holds:*
(i) $\widetilde{P} = \widetilde{Q} \in \{\widetilde{I}, \widetilde{T}, \widetilde{R}, \widetilde{RTR}\}$ (i.e., $\widetilde{P} = \widetilde{Q}$ is an involution),
(ii) $\{\widetilde{P}, \widetilde{Q}\} = \{\widetilde{TR}, \widetilde{RT}\}$.

Note that $T(w_2) = w_3$, $T(w_3) = w_2$, and $T(w_1) = w_1$. Also, for $n = 4$, note that $Rw_1 = -w_3, Rw_2 = -w_2$, and since $-I \in \mathbf{D}_4$, we conclude that w_1 and w_3 can be transformed one into another by an operator $-R$ from $\mathbf{F}_4 \setminus \mathbf{D}_4$, which fixes w_2. Therefore the group $\mathbf{F}_4$ transitively acts on the set $\mathrm{Orb}_{\mathbf{D}_4} w_1 \cup \mathrm{Orb}_{\mathbf{D}_4} w_2 \cup \mathrm{Orb}_{\mathbf{D}_4} w_3$. This, together with Lemmas 5.2, 5.3, implies the following result:

Lemma 5.4. *The action of the group $\mathcal{RE}(\mathbf{D}_n)$ on the set $\mathcal{B}(\mathbf{D}_n)$ is transitive. In particular:*

(a) *if* $P,Q \in \mathbf{D}_n$*, then* $P\mathbf{S}_iQ = \mathbf{S}_i$*,* $P\mathbf{S}_i^tQ = \mathbf{S}_i^t$ *for* $i = 1,2,3$.
(b) *if* $P,Q \in \mathbf{B}_n\backslash\mathbf{D}_n$*, then* $PQ \in \mathbf{D}_n$*, and* $P\mathbf{S}_1Q = \mathbf{S}_2$*,* $P\mathbf{S}_2Q = \mathbf{S}_1$*,* $P\mathbf{S}_3Q = \mathbf{S}_3^t$*,* $P\mathbf{S}_1^tQ = \mathbf{S}_2^t$*,* $P\mathbf{S}_2^tQ = \mathbf{S}_1^t$*,* $P\mathbf{S}_3^tQ = \mathbf{S}_3$*.*
(c) *if* $n = 4$ *then for* $P,Q \in \widetilde{R}$ *we have* $P\mathbf{S}_1Q = \mathbf{S}_3$*,* $P\mathbf{S}_2Q = \mathbf{S}_2^t$*,* $P\mathbf{S}_3Q = \mathbf{S}_1^t$*,* $P\mathbf{S}_1^tQ = \mathbf{S}_3$*,* $P\mathbf{S}_2^tQ = \mathbf{S}_2$*,* $P\mathbf{S}_3^tQ = \mathbf{S}_1$*.*

Proof. Direct verification. ■

Theorem 5.5. *Let* $n \geq 4$*. Then*

$$\mathcal{L}(\mathbf{D}_n) = \mathcal{L}(\mathcal{B}(\mathbf{D}_n)) = \mathcal{RE}(\mathbf{D}_n).$$

In other words, the following conditions are equivalent for a linear transformation ϕ *of* $\mathbf{M}_n(\mathbb{R})$*.*

(a) $\phi(\mathbf{D}_n) = \mathbf{D}_n$*.*
(b) $\phi(\mathcal{B}(\mathbf{D}_n)) = \mathcal{B}(\mathbf{D}_n)$*, equivalently,* $\phi(\operatorname{env}\mathbf{D}_n) = \operatorname{env}\mathbf{D}_n$*.*
(c) *There exist* P,Q *belonging to* $\mathbf{B}_n$*, if* $n \geq 5$*, or to* $\mathbf{F}_4$*, if* $n = 4$*, satisfying* $PQ \in \mathbf{D}_n$ *such that* ϕ *is of the form*

$$A \mapsto PAQ \quad \text{or} \quad A \mapsto PA^tQ.$$

Proof. The implication (c) ⇒ (a) follows from Lemma 1.1. The implication (c) ⇒ (b) follows from Lemma 5.4 and the fact that $\mathcal{B}(\mathbf{D}_n)$ is invariant under the transposition.

(a) ⇒ (c). Let ϕ belong to $\mathcal{L}(\mathbf{D}_n)$. Then ϕ preserves the inner product on $\mathbf{M}_n(\mathbb{R})$. Also, we may assume that $\phi(I) = I$. With this assumption, we show that there exists $P \in \mathbf{B}_n$ (or $P \in \mathbf{F}_4$ when $n = 4$) such that ϕ is of the form $A \mapsto P^tAP$ or $A \mapsto P^tA^tP$.

Let $\mathbf{R}_1$ consist of the matrices $F_{ij} = I - (e_i - e_j)(e_i - e_j)^t$ for $1 \leq i < j \leq n$, and $\mathbf{R}_1'$ consist of the matrices $F_{ij}' = I - (e_i + e_j)(e_i + e_j)^t$ for $1 \leq i < j \leq n$.

Claim 1. $\{X \in \mathbf{D}_n : (I,X) = n-2\} = \mathbf{R}_1 \cup \mathbf{R}_1'$.

Thus, $\phi(\mathbf{R}_1 \cup \mathbf{R}_1') = \mathbf{R}_1 \cup \mathbf{R}_1'$.

We may assume that $\phi(F_{12}) = F_{12}$; otherwise, replace ϕ by a mapping of the form $A \mapsto P_1^t\phi(A)P_1$ for a suitable $P_1 \in \mathbf{B}_n$.

Claim 2. $X \in \mathbf{R}_1 \cup \mathbf{R}_1'$ *satisfies* $(F_{12},X) = n-3$ *if and only if* $X = F_{ij}$ *or* F_{ij}' *with* (i) $i = 1$ *and* $3 \leq j \leq n$*, or* (ii) $i = 2$ *and* $3 \leq j \leq n$*.*

So, $\phi(F_{13})$ equals either F_{ij} or F_{ij}', with i,j as in (i) or (ii). In case (i), replace ϕ by a mapping $A \mapsto P_2^t\phi(A)P_2$, where P_2 is an appropriately signed $(3,j)$ transposition, to fix F_{13}. A $(1,2)$ transposition P_3 can be used to reduce case (ii) to case (i). Note that the property $\phi(F_{12}) = F_{12}$ is preserved under these changes. Thus, we may assume that $\phi(F_{1j}) = F_{1j}$ simultaneously for $j = 2$ and $j = 3$.

Now, consider F_{1j} for $j = 4, \dots, n$. Due to the orthogonality of ϕ, $X_j = \phi(F_{1j})$ is such that $(F_{12}, X_j) = (F_{13}, X_j) = n - 3$.

Claim 3. $\{X \in \mathbf{R}_1 \cup \mathbf{R}'_1 \colon (F_{12}, X) = (F_{13}, X) = n-3\} = \{F_{1,s}, F'_{1,s} \colon s = 4, \dots, n\}$.

Thus, $X_j = F_{1,s_j}$ or $X_j = F'_{1,s_j}$ for some $s_j \in \{4, \dots, n\}$. But then (similarly to case (i) for $j = 3$ above) the transformation ϕ can be replaced by $A \mapsto P_j^t \phi(A) P_j$ (with P_j being a suitably signed (j, s_j) transposition) in such a way that F_{1j} becomes a fixed point of ϕ. Since these replacements do not change the values $\phi(F_{1k})$ for $k < j$, implementing them consequently we achieve the property

$$\phi(F_{ij}) = F_{ij}; \qquad j = i+1, \dots, n \tag{10}$$

for $i = 1$.

Denote by B_{ijk} the $n \times n$ permutation matrices corresponding to the 3-cycles (ijk). Let $\mathbf{R}_2 (\subset \mathbf{D}_n)$ stand for the set of all such matrices, and consider $X = \phi(B_{ijk})$. Due to (10),

$$(F_{ij}, X) = (F_{ik}, X) = n - 2 \tag{11}$$

for $i = 1$.

Claim 4. *For $j \neq k$ distinct form 1,*

$$\{X \in \mathbf{D}_n \colon (I, X) = n-3, (F_{1j}, X) = (F_{1k}, X) = n-2\} = \{B_{1jk}, B_{1kj}\}.$$

We may assume that

$$\phi(B_{123}) = B_{123}; \tag{12}$$

otherwise, replace ϕ by a mapping of the form $A \mapsto \phi(A)^t$. For $k > 3$, the scalar product $(B_{123}, B_{12k}) = n - 3$ is different from $(B_{123}, B_{1k2}) = n-4$. Thus, condition (12) automatically implies that $\phi(B_{12k}) = B_{12k}$, $k = 3, \dots, n$. Observe now that $(B_{12k}, B_{1jk}) = n - 3$, $(B_{12k}, B_{1kj}) = n - 4$ for $j, k \neq 2$. Hence, $\phi(B_{1jk})$ must be different from B_{1kj}. The only remaining option is

$$\phi(B_{ijk}) = B_{ijk}; \quad j, k = i+1, \dots, n, \ j \neq k \tag{13}$$

for $i = 1$. Note also that all the matrices F_{ij} are symmetric so that property (10) for $i = 1$ still holds.

We now return to matrices F_{ij}, with arbitrary i.

Claim 5. *For $i \neq j$ and distinct form 1,*

$$\{X \in \mathbf{R}_1 \cup \mathbf{R}'_1 \colon (F_{1i}, X) = (F_{1j}, X) = n-3, (B_{1ij}, X) = n-2\} = \{F_{ij}\}.$$

Applying Claim 5 to $\phi(F_{2j}) = X$, we conclude that (10) holds for all i. But then (11) along with

$$(B_{1jk}, B_{ijk}) = n - 3 \neq (B_{1jk}, B_{ikj}) = n - 4$$

(for $1, i, j, k$ being mutually distinct) show that (13) holds for all admissible i. In other words,

$$\phi(X) = X \text{ for all } X \in \mathbf{R}_1 \cup \mathbf{R}_2. \tag{14}$$

Since $\phi(\mathbf{R}_1 \cup \mathbf{R}_1') = \mathbf{R}_1 \cup \mathbf{R}_1'$, we conclude from (14) that in particular $\phi(\mathbf{R}_1') = \mathbf{R}_1'$. Observe that

$$(F_{ij}', F_{pq}) = \begin{cases} n-4 & \text{if } \{p,q\} \cap \{i,j\} = \emptyset \text{ or } \{p,q\} = \{i,j\} \\ n-3 & \text{otherwise} \end{cases}.$$

If $n > 4$, then for any two different pairs $\{i,j\}$ and $\{i',j'\}$ it is possible to find $\{p,q\}$ disjoint with $\{i,j\}$ and having exactly one common element with $\{i',j'\}$. Since ϕ preserves inner product and elements of $\mathbf{R}_1$, the property $\phi(\mathbf{R}_1') = \mathbf{R}_1'$ can be strengthened to

$$\phi(X) = X \text{ for all } X \in \mathbf{R}_1'. \tag{15}$$

For $n = 4$, we can at this point derive only a weaker conclusion

$$\phi(F_{ij}') = F_{ij}' \text{ or } \phi(F_{ij}') = F_{i',j'}'$$

with $\{i',j'\}$ being a complement of $\{i,j\}$ to $\{1,2,3,4\}$. However, in this case we have yet another degree of freedom at our disposal, namely, the replacement of ϕ by $A \mapsto R\phi(A)R$ with R given by (8). It is easy to check that $RXR = X$ for all $X \in \mathbf{R}_1 \cup \mathbf{R}_2$ and $RF_{ij}'R = F_{i',j'}'$. Hence, by using this replacement we can fix one of the elements of $\mathbf{R}_1'$ while keeping (14) valid. Suppose therefore that $\phi(F_{12}') = F_{12}'$. Then of course $\phi(F_{34}') = F_{34}'$ as well. To prove (15) in the case $n = 4$, it remains to show that

$$\phi(F_{13}') = F_{13}', \quad \phi(F_{14}') = F_{14}'. \tag{16}$$

To this end, we introduce the set $\mathbf{R}_2'$ of all signed permutation matrices B_{ijk}' corresponding to 3-cycles (ijk) and having -1 in the positions (ij) and (jk).

Claim 6. $\mathbf{R}_2 \cup \mathbf{R}_2' = \{X \in \mathbf{D}_n \colon (I, X) = n - 3\}$.

Hence, property (14) implies $\phi(\mathbf{R}_2') = \mathbf{R}_2'$. Observe now that, for $n = 4$, $(F_{pq}, B_{ijk}') = 2$ if and only if $\{p,q\} = \{i,k\}$. Hence, $\phi(B_{ijk}')$ is either B_{ijk}' itself or B_{isk}', where $s \in \{1,2,3,4\}$ is different from i,j,k. But

$$(B_{123}', F_{12}') = (B_{124}', F_{12}') = 2 \neq 0 = (B_{134}', F_{12}') = (B_{143}', F_{12}').$$

Thus, $\phi(F'_{12}) = F'_{12}$ implies that $\phi(B'_{123}) = B'_{123}$, $\phi(B'_{124}) = B'_{124}$. Now (16) follows from yet another simple computation showing that

$$(B'_{124}, F'_{13}) = (B'_{123}, F'_{14}) = 0 \neq 2 = (B'_{124}, F'_{24}) = (B'_{123}, F'_{23}).$$

So, both for $n = 4$ and $n > 4$ we may suppose that (15) holds along with (14). Since the set $\mathbf{R}_1 \cup \mathbf{R}'_1 \cup \mathbf{R}_2$ spans $\mathbf{M}_n(\mathbb{R})$, the only linear transformation ϕ satisfying (14), (15) is the identity transformation.

Next, we turn to (b) $\Rightarrow$ (c). Let w_j, $\mathbf{S}_j$, and $\mathbf{S}_j^t$ ($j = 1, 2, 3$) be defined as before.

First we show that (probably, after some modifications, replacing ϕ by the mapping

$$A \mapsto P\phi(A)Q \text{ or } A \mapsto XP\phi(A)^tQ, \tag{17}$$

P, Q in the normalizer of $\mathbf{D}_n$, $PQ \in \mathbf{D}_n$) we may suppose that

$$\phi(\mathbf{S}_3 \cup \mathbf{S}_3^t) = \mathbf{S}_3 \cup \mathbf{S}_3^t, \tag{18}$$

and

$$\phi(A_1) = A_1, \text{ where } A_1 = w_1 w_2^t. \tag{19}$$

If $n > 4$, property (18) holds (with no modifications of ϕ needed), because of the following

Claim 7. $\mathbf{S}_3 \cup \mathbf{S}_3^t$ *consists exactly of all* $A \in \mathcal{B}(\mathbf{D}_n)$ *for which there exist* $B \in \mathcal{B}(\mathbf{D}_n)$ *such that*

$$(A, B) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

Since ϕ preserves the inner product and leaves $\mathcal{B}(\mathbf{D}_n)$ invariant, it therefore must leave $\mathbf{S}_3 \cup \mathbf{S}_3^t$ invariant as well. From (18) we conclude that also

$$\phi(\mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_1^t \cup \mathbf{S}_2^t) = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_1^t \cup \mathbf{S}_2^t. \tag{20}$$

In particular, $X_1 = \phi(A_1)$ is a matrix of the form $Pw_1w_i^tQ$ or $Pw_iw_1^tQ$ for $i = 2, 3$ and $P, Q \in \mathbf{D}_n$. We can then replace ϕ by a mapping of the form (17) with $P, Q \in \mathbf{B}_n$ so that $PQ \in \mathbf{D}_n$ and the resulting mapping fixes A_1.

If $n = 4$, property (18) does not necessarily hold for the original mapping ϕ, and $X_1 = \phi(A_1)$ can be any element of $2\mathcal{B}(\mathbf{D}_4)$. However, if $X_1 \in \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_1^t \cup \mathbf{S}_2^t$, the same reasoning as above can be applied to achieve (19). Consider the remaining possibility $X_1 = Pw_2w_3^tQ$ or $X = Pw_3w_2^tQ$ for some $P, Q \in \mathbf{D}_4$. We may assume that $X_1 = w_3w_2^t$ by a suitable modification of ϕ of the form (17). Now replace ϕ by a mapping of the form $A \mapsto R\phi(A)R$, with R given by (8). One can check that the resulting map will send $w_1w_2^t$ to matrix of the form $Xw_1w_2^tY$ for some $X, Y \in \mathbf{D}_4$. Hence, we are back to the previous case, and we can further modify ϕ so that the resulting map will again fix A_1. So, without loss of generality we may suppose that (19) holds.

We will now make use of

Claim 8. $\{Y \in 2\mathcal{B}(\mathbf{D}_4) \colon (A_1, Y) = 8\} = \{B_j = w_1(e - 2e_j)^t \colon j = 1, \dots, 4\}$.

Hence, condition (19) guarantees that ϕ maps the set $\{B_j\}_{j=1}^4$ onto itself. Replacing ϕ by $X \mapsto \phi(X)P$ with an appropriate permutation P, we may achieve the property $\phi(B_j) = B_j$ $(j = 1, \dots, 4)$ while still having (19). But then (18) holds as well, because of

Claim 9. $\mathbf{S}_3 \cup \mathbf{S}_3^t = \{Y \in 2\mathcal{B}(\mathbf{D}_4) \colon Y \perp \{A_1, B_1, B_2, B_3, B_4\}\}$.

So, both for $n = 4$ and for $n > 4$ we may achieve (18) and (19).

Since (20) holds along with (18), the mapping ϕ satisfying this property is a $\mathcal{B}(\mathbf{B}_n)$ preserver. Due to Theorem 4.1, it has the form (2). In the next step of the proof we show that the Schur multiplier S in this representation can be eliminated. It suffices to consider the case of the first formula in (2); the second one can then be covered by passing from ϕ to ϕ^t.

We use the standard notation $X[ij]$ for the 2×2 block $\begin{bmatrix} x_{ii} & x_{ij} \\ x_{ji} & x_{jj} \end{bmatrix}$ of an $n \times n$ matrix X. Suppose that the rank of S is at least two. Then there exist i, j such that $\det S[ij] \neq 0$. But then $\det(S \circ (w_2w_3^t)[ij]) \neq 0$, that is, the rank of the matrix $S \circ (w_2w_3^t)$ is also at least two. Since $S \circ (w_2w_3^t)$ is in the image of $\mathbf{S}_3 \cup \mathbf{S}_3^t$ under the mapping (2), this is a contradiction with (18). Hence, S is a rank one matrix, that is, $S = uv^t$, where $u = (u_1, \dots, u_n)^t$, $v = (v_1, \dots, v_n)^t$ and $u_j, v_j = \pm 1$ for $j = 1, \dots, n$. It remains to observe that $(uv^t) \circ (PZQ) = P_1ZQ_1$, where

$$P_1 = \operatorname{diag}[u_1, \dots, u_n]P, \; Q_1 = Q \operatorname{diag}[v_1, \dots, v_n] \in \mathbf{B}_n.$$

The last step is to show that $P_1Q_1 \in \mathbf{D}_n$. To this end, recall property (19), according to which $P_1A_1Q_1 = A_1$. This is only possible if the signs of all non-zero entries of Q_1 are the same. Without loss of generality (changing the sign of both P_1 and Q_1 if necessary) we may suppose that Q_1 is a permutation matrix. Consider then $P_1w_2w_3^tQ_1$. This matrix lies in $\mathbf{S}_3 \cup \mathbf{S}_3^t$ due to (18). But this is only possible if $P_1 \in \mathbf{D}_n$. Then $P_1Q_1 \in \mathbf{D}_n$ as well. ■

6. Groups $\mathbf{I}_2(n)$

Matrix Realization

Group $\mathbf{I}_2(n)$ is the dihedral group, i.e., the group of symmetries of a regular n-side convex polygon. So, we always assume that $n \geq 3$. Assuming that a vertex of this n-side convex polygon is on the positive x-semi-axis, we arrive to the following matrix realization.

Let

$$R_n = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathbf{I}_2(n) = \mathrm{Rot}_n \cup \mathrm{Refl}_n$, where

$$\mathrm{Rot}_n = \{R_n^k : 0 \leq k < n\}$$

is the set of rotations in $\mathbf{I}_2(n)$, and

$$\mathrm{Refl}_n = \{DX : X \in \mathrm{Rot}_n\} = \{R_{2n}^k D R_{2n}^{-k} : 0 \leq k < n\}$$

is the set of reflections in $\mathbf{I}_2(n)$.

Birkhoff Tensors

Let $w_1 = (1,0)^t$, $w_2 = (\cos\pi/n, \sin\pi/n)^t$. Then

$$(\cos\frac{\pi}{n})\mathcal{B}(\mathbf{I}_2(n)) = \left\{Pw_1w_2^tQ,\ Qw_2w_1^tP \colon P, Q \in \mathbf{I}_2(n)\right\}.$$

Possible inner products

Let $X \in \mathbf{I}_2(n)$, $X \neq I$. Then

$$(I, X) \in \{2\cos\frac{2k\pi}{n} \colon k = 1, \ldots, n-1\}.$$

Normalizers

Lemma 6.1. *The normalizer $N(\mathbf{I}_2(n))$ of $\mathbf{I}_2(n)$ in $O(2)$ coincides with $\mathbf{I}_2(2n)$.*

Proof. Rot_{2n} and D are obviously in $N(\mathbf{I}_2(n))$, the reflection D and rotations from Rot_{2n} generate the whole $\mathbf{I}_2(2n)$, so $\mathbf{I}_2(2n) \subset N(\mathbf{I}_2(n))$. Every operator from $O(V)$ is either a rotation R or a reflection DR. If a rotation R is in $N(\mathbf{I}_2(n))$ then $RDR^* \in \mathbf{I}_2(n)$, therefore $R \in \mathrm{Rot}_{2n}$. If a reflection DR is in $N(\mathbf{I}_2(n))$ then $(DR)^*D(DR) \in \mathbf{I}_2(n)$, therefore $R \in \mathbf{I}_2(2n)$. So, $N(\mathbf{I}_2(n)) = \mathbf{I}_2(2n)$. ∎

Linear Preservers

Theorem 6.2. *Let $n \geq 3$. Then*

$$\mathcal{L}(\mathbf{I}_2(n)) = \mathcal{L}(\mathcal{B}(\mathbf{I}_2(n))) = \mathcal{RE}(\mathbf{I}_2(n)).$$

In other words, the following conditions are equivalent for a linear transformation ϕ of $M_2(\mathbb{R})$:

(a) $\phi(\mathbf{I}_2(n)) = \mathbf{I}_2(n)$.
(b) $\phi(\mathcal{B}(\mathbf{I}_2(n))) = \mathcal{B}(\mathbf{I}_2(n))$.
(c) *There exist $P, Q \in \mathbf{I}_2(2n)$ satisfying $PQ \in \mathbf{I}_2(n)$ such that ϕ is of the form*

$$A \mapsto PAQ \quad \text{or} \quad A \mapsto PA^tQ.$$

Proof. By (1), (a) and (b) are equivalent. Clearly, (c) implies (a). It remains to prove that (a) implies (c).

By Lemma 2.1, if $\phi(\mathbf{I}_2(n)) = \mathbf{I}_2(n)$, then ϕ preserves the inner product on $M_2(\mathbb{R})$. We may assume that $\phi(I) = I$.

Claim 1. $\{X \in \mathbf{I}_2(n)\colon (I, X) = 2\cos(2\pi/n)\} = \{R_n, R_n^t\}$.

We may assume that $\phi(R_n) = R_n$; otherwise, replace ϕ by the mapping $A \mapsto \phi(A)^t$. It follows from Claim 1 that $X \in \mathbf{I}_2(n)$ satisfies $(R_n, X) = (R_n, R_n^2)$ if and only if $X = R_n^2$ or $X = I$. Since $\phi(I) = I, \phi(R_n) = R_n$ then $\phi(R_n^2) = R_n^2$. Similarly, we get that ϕ fixes the whole set Rot_n. Therefore, $\phi(\mathrm{Refl}_n) = \mathrm{Refl}_n$.

The transformations $X \mapsto RXR^t$ for $R \in \mathrm{Rot}_{2n}$ fix every element of Rot_n, so we may assume that $\phi(D) = D$. Applying Claim 1 once again, we observe that $X \in \mathrm{Refl}_n$ satisfies $(D, X) = (D, DR_n)$ if and only if $X = DR_n$ or $X = DR_n^{n-1}$. Invoking the transformation $X \mapsto R_nXR_n^t$ if necessary, we may assume that $\phi(DR_n) = DR_n$.

Now, using the fact that $(I, X) = (I, \phi(X))$ and $(R_n, X) = (R_n, \phi(X))$ for each $X \in \mathbf{I}_2(n)$, we see that the modified ϕ satisfies $\phi(X) = X$ for all $X \in \mathbf{I}_2(n)$. The result follows. ∎

7. Final remarks

It would be interesting to figure out the structure of $\mathcal{L}(G)$ for exceptional finite irreducible Coxeter groups. The cases $G = \mathbf{F}_4$ and $\mathbf{H}_3$ can be handled by methods similar to those used in this paper. The case $G = \mathbf{H}_4$ has some complication, namely, its Coxeter graph is non-branching, but it has not yet been proven (though seems very probable) that $\mathrm{Extr}\, G^\circ = \mathcal{B}(G)$.

The cases of the other three exceptional groups with branching graphs — $G = \mathbf{E}_6, \mathbf{E}_7$, and $\mathbf{E}_8$ — seem to be much more difficult. In these cases $\mathrm{Extr}\, G^\circ \neq \mathcal{B}(G)$, and it is possible that $\mathcal{L}(G)$ differs from $\mathcal{L}(\mathcal{B}(G))$. It is very unlikely that the group $\mathcal{RE}(G)$ acts transitively on $\mathcal{B}(G)$ in these cases.

Our proofs in this paper are rather computational. It would be nice to find a unified approach.

References

[1] Auerbach, H., Sur les groupes bornés de substitutions linéaires, *C.R. Acad. Sci. Paris* **195** (1932), 1367–1369.

[2] Benson, C., and Grove, L., *Finite Reflection Groups*, 2nd ed., Springer-Verlag, New York, 1985.

[3] Birkhoff, G., Tres observaciones sobre el algebra lineal, *University Nac. Tucuman Rev. Ser.* **A5** (1946), 147–150.

[4] Bourbaki, N., *Groupes et Algebres de Lie*, Ch. IV–VI, Hermann, Paris, 1968.

[5] Brandman, J., Fowler, J., Lins, B., Spitkovsky, I., and Zobin, N., Convex hulls of Coxeter groups, 22 pp., submitted; preprint available from http://www.math.wm.edu/~zobin/.

[6] Deutsch, E., and Schneider, H., Bounded groups and norm-hermitian matrices, *Linear Algebra and Appl.* **9** (1974), 9–27.

[7] Dixon, J., Rigid embedding of simple groups in the general linear group, *Canad. J. Math.* **29** (1977), no. 2, 384–391.

[8] Djokovic, D.Z., Li, C.K., and Rodman, L., Isometries of symmetric gauge functions, *Linear and Multilin. Algebra* **30** (1991), 81–92.

[9] Gohberg, I.C., and Krein, M.G., *Introduction to the Theory of Linear Nonselfadjoint operators*, Transl. Math. Monogr., vol.18, Amer. Math. Soc., Providence, R.I., 1969.

[10] Li, C.K., Some aspects of the theory of norms, *Linear Algebra and Appl.* **212** (1994), 71–100.

[11] Li, C.K., Tam, B.S., and Tsing, N.K., Linear maps preserving permutation and stochastic matrices, to appear in the *Linear Algebra and Appl.* (2001), preprint available at http://www.wm.edu/CAS/MINEQ/pszips/tsing.ps.gz

[12] Marcus, M., All linear operators leaving the unitary group invariant, *Duke Math. Journal* **26** (1959), 155–163.

[13] McCarthy, N., Ogilvie, D., Spitkovsky, I., and Zobin, N., Birkhoff's Theorem and convex hulls of Coxeter groups, 10 pp., submitted; preprint available at http://www.math.wm.edu/~zobin/.

[14] Pierce, S., et al., A Survey of Linear Preserver Problems, *Linear and Multilin. Algebra* **33** (1992), 1–130.

[15] Wei, A., Linear transformations preserving the real orthogonal group, *Canad. Journal of Math.* **27** (1975), 561–572.

[16] Zobin, N., and Zobina, V., Coxeter groups and interpolation of operators, *Integral Eq. and Operator Theory* **18** (1994), 335–367.

[17] Zobin, N., and Zobina, V., A general theory of sufficient collections of norms with a prescribed semigroup of contractions, *Operator Theory: Advances and Appl.* **73** (1994), 397–416.