ON EIGENVALUES AND BOUNDARY CURVATURE OF THE NUMERICAL RANGE

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Abstract. For an $n\times n$ matrix A, let $M(A)$ be the smallest possible constant in the inequality $D_p(A) \le M(A)R_p(A)$. Here p is a point on the smooth portion of the boundary $\partial W(A)$ of the numerical range of A, $R_p(A)$ is the radius of curvature of $\partial W(A)$ at this point, and $D_p(A)$ is the distance from p to the spectrum of A. We show that $M(A) \le n/2$ and that there exist A with $M(A) \ge \frac{n}{2}\sin\frac{\pi}{n}$. We also describe a class of matrices with $M(A) \le 1$ (including, of course, all 2×2 matrices).

1. Introduction

Let A be an $n\times n$ matrix with complex entries: $A \in \mathbb{C}^{n\times n}$. The *numerical range* of A is defined as $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n,\ \|x\| = 1\}$, where $\langle .,.\rangle$ and $\|.\|$ are the standard scalar product and norm on $\mathbb{C}^n$, respectively. There is an extensive literature on the properties of $W(A)$, starting with the classical papers by Toeplitz [14] and Hausdorff [4]. All the unreferenced properties of the numerical range in this paper can be found in Chapter 1 of [5]; see also [3].

It is well known that $W(A)$ is a convex compact subset of $\mathbb{C}$ (containing the spectrum $\sigma(A)$ of A) with a piecewise analytic boundary $\partial W(A)$. Hence, for all but finitely many points $p \in \partial W(A)$, the radius of curvature $R_p(A)$ of $\partial W(A)$ at p is well-defined. By convention, $R_p(A) = 0$ if p is a corner point of $W(A)$, and $R_p(A) = \infty$ if p lies inside a flat portion of $\partial W(A)$.

Let $D_p(A)$ denote the distance from p to $\sigma(A)$, and let $M(A)$ be the smallest constant such that

$$D_p(A) \le M(A)R_p(A) \tag{1}$$

for all $p \in \partial W(A)$ where $R_p(A)$ is defined. By Donoghue's theorem, $D_p(A) = 0$ whenever $R_p(A) = 0$. Therefore, $M(A) = 0$ for all *convexoid* matrices A, that is, for matrices with polygonal numerical ranges. For non-convexoid A,

$$M(A) = \sup \frac{D_p(A)}{R_p(A)}$$

where the supremum in the right hand side is taken along all points $p \in \partial W(A)$ with finite non-zero curvature.

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Computation of $M(A)$ for arbitrary A is an interesting open problem. In this paper, we find upper and lower bounds for

$$M_n = \sup\{M(A)\colon A \in \mathbb{C}^{n\times n}\},$$

namely,

$$\frac{n}{2}\sin\frac{\pi}{n} \leq M_n \leq \frac{n}{2}. \tag{2}$$

Section 3 contains the proof of the upper bound in (2).This proof rests on a number of auxiliary results, found in Section 2. We believe that some of these results may be of independent interest.

For $n = 2$, the upper and lower bounds in (2) coincide, so that $M_2 = 1$. This value of $M(A)$ is assumed on 2×2 matrices A with circular $W(A)$, that is, on non-normal A with coinciding eigenvalues. In Section 4, we give a description of some higher dimensional matrices A where $M(A) = 1$, as well as some elementary computations of the exact value of $M(A)$ for all 2×2 matrices A. Such computations provide an alternative proof of the equality $M_2 = 1$. In the last section, we derive explicit formulas for $D_p(A)$ and $R_p(A)$ for some unicellular $n \times n$ matrices A. We use these formulas to obtain the lower bound in (2). As a byproduct, the value of $M(A)$ is computed for a unicellular 3×3 matrix A with a flat portion on the boundary of its numerical range.

For $n \geq 3$, we do not have an exact value M_n. In fact, it is not even clear whether a sequence M_n is bounded. The question whether there exists a universal constant M such that

$$D_p(A) \leq MR_p(A) \text{ for all square matrices } A$$

remains open. This question, posed by Roy Mathias in January of 1997 (see the Matrix Inequalities in Science and Engineering web page
http://www.wm.edu/CAS/MINEQ/topics/970103.html), served as a starting point for this research. If such a constant M exists, it follows from (2) that its value cannot be smaller than $\pi/2$.

Throughout the paper, we will use the standard notation $X_R = \frac{1}{2}(X + X^*)$ and $X_J = \frac{1}{2i}(X - X^*)$ for the real and imaginary part of any square matrix X. We denote the (j, k)-entry of X by X_{jk}; the matrix obtained from X by deleting its j-th row and k-th column by $X[jk]$; the transposed matrix of X by X^T; and the upper half plane $\{z \in \mathbb{C}\colon \operatorname{Im} z \geq 0\}$ by $\mathbb{C}_+$.

2. Auxiliary results

Recall that a matrix A is *unitarily reducible* if it is unitarily similar to a direct sum $A_1 \oplus \cdots \oplus A_k$ of (smaller in size) matrices $A_1, \ldots, A_k$, $k \geq 2$:

$$A = U^*(A_1 \oplus \cdots \oplus A_k)U \tag{3}$$

for some unitary matrix U.

Lemma 2.1. *Under the condition* (3), $M(A) \leq \max_{1\leq j\leq k} M(A_j)$.

Proof. The numerical range of A is the convex hull of the numerical ranges of the blocks A_j:

$$W(A) = \operatorname{conv}\{W(A_1), \ldots, W(A_k)\}.$$

Hence, $\partial W(A)$ consists of portions of $\partial W(A_j)$ connected by the straight line segments. It remains to observe that, for $p \in \partial W(A_j) \cap \partial W(A)$,

$$\operatorname{dist}(p, \sigma(A)) \le \operatorname{dist}(p, \sigma(A_j)) \le M(A_j) R_p(A_j) \le M(A_j) R_p(A).$$

■

The result of Lemma 2.1 is not sharp. For example, a general convexoid matrix A is unitarily similar to a direct sum of a normal matrix A_1 with an arbitrary matrix A_2 such that $W(A_2) \subset W(A_1)$. In this case $M(A) = M(A_1) = 0$ while $M(A_2)$ can be positive.

Lemma 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be such that $0 \in \partial W(A)$ and $W(A)$ lies entirely in the upper half plane. Then A is unitarily similar to a matrix of the form*

$$\begin{bmatrix} 0 & \epsilon & 0 & \dots & 0 \\ \epsilon & & & & \\ 0 & & & & \\ \vdots & & B & & \\ 0 & & & & \end{bmatrix}, \tag{4}$$

where $\epsilon \ge 0$ and B is an $(n-1)\times(n-1)$ matrix with $B_J \ge 0$.

Proof. Choose a unit vector $e_1 \in \mathbb{C}^n$ such that $\langle Ae_1, e_1\rangle = 0$; this is possible since $0 \in W(A)$. Let $e_2 = \|Ae_1\|^{-1} Ae_1$ if $Ae_1 \ne 0$, or an arbitrary unit vector orthogonal to e_1 otherwise. Then extend $\{e_1, e_2\}$ to an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$. The matrix C with the entries $C_{jk} = \langle Ae_k, e_j\rangle$ $(j, k = 1, \dots, n)$ is unitarily similar to A. Since $\langle Ae_1, e_j\rangle = \|Ae_1\|\langle e_2, e_j\rangle$, the first column of C is indeed as in (4), with $\epsilon = \|Ae_1\| \ge 0$. As was shown in [6, Lemma 3.1], if x is a vector such that $\langle Ax, x\rangle = 0 \in \partial W(A)$ and y is any vector perpendicular to x, then $\langle Ax, y\rangle = \overline{\langle Ay, x\rangle}$. Letting $x = e_1$ and $y = e_j$ $(j \ne 1)$ one at a time, we see that $\langle Ae_j, e_1\rangle = \langle Ae_1, e_j\rangle$. In other words, the first row of C also is as in (4).

Finally, the numerical range of the matrix $B = C[11]$ lies in $W(C) = W(A)$, and therefore in $\mathbb{C}_+$. This condition is equivalent to B_J being non-negative. ■

Observe (though we will not use this) that the converse to Lemma 2.2 is also true: if C has the form (4), then $C_J = \{0\} \oplus B_J$, so that $C_J \ge 0$ and $W(C) = W(A)$ lies in $\mathbb{C}_+$. On the other hand, any diagonal entry of C lies in $W(C)$, so that $0 = C_{11} \in W(A)$.

If $\epsilon > 0$ and $B_J > 0$, then the radius of curvature $R_0(A)$ can be computed using the following Fiedler's result [1].

Lemma 2.3. *Let $A \in \mathbb{C}^{n\times n}$, and let z be a unit vector corresponding to a boundary point $p = \langle Az, z\rangle$ of $W(A)$. Also let $ux+vy+w = 0$ be an equation of the supporting line of $W(A)$ at the point p. If $-w$ is a simple eigenvalue of $P = uA_R + vA_J$, then $\partial W(A)$ is smooth in the neighborhood of p, and its radius of curvature at this point equals*

$$R_p(A) = \frac{2}{\sqrt{u^2+v^2}} |\langle (P+wI)^+ Qz, Qz\rangle|. \tag{5}$$

Here $Q = vA_R - uA_J$, and X^+ stands for the Moore-Penrose inverse of X.

For the matrix (4) one may choose $u = 0$, $v = 1$, $w = 0$ to obtain $P + wI = A_J$, $Q = A_R$. Moreover, $z = [1, 0 \dots, 0]^T$, and therefore $Qz = [0, \epsilon, 0, \dots, 0]^T$. If B_J is strictly positive, then zero is a simple eigenvalue of A_J, its Moore-Penrose inverse is $(A_J)^+ = 0 \oplus (B_J)^{-1}$, and formula (5) yields

$$R_0(A) = [0, \epsilon, 0, \ldots, 0] \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & & & & \\ 0 & & (B_J)^{-1} & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \begin{bmatrix} 0 \\ \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 2\epsilon^2 (B_J^{-1})_{11}.$$

Hence, the next result:

Lemma 2.4. *Let A be of the form* (4)*, with $\epsilon > 0$ and $B_J > 0$. Then the origin lies on the smooth portion of $\partial W(A)$, and*

$$R_0(A) = 2\epsilon^2 (B_J^{-1})_{11} = 2\epsilon^2 \frac{\det B_J[11]}{\det B_J}. \tag{6}$$

We will use (6) to find the upper bound for $D_0(A)/R_0(A)$ when A is of the form (4) with $\epsilon > 0$ and $B_J > 0$. Before we do this, we need two additional auxiliary results.

Lemma 2.5. *Let $X \in \mathbb{C}^{n\times n}$ be such that $X_R > 0$. Then $|(X^{-1})_{11}| \le (X_R^{-1})_{11}$.*

Proof. Rewrite $X = X_R + iX_J$ as

$$X = X_R^{1/2}(I + iX_R^{-1/2} X_J X_R^{-1/2}) X_R^{1/2},$$

where $X_R^{1/2}$ is the positive square root of X_R. Then $X^{-1} = X_R^{-1/2} Y X_R^{-1/2}$, where $Y = (I + iX_R^{-1/2} X_J X_R^{-1/2})^{-1}$, and for any non-zero $f \in \mathbb{C}^n$:

$$\frac{\langle X^{-1}f, f\rangle}{\langle X_R^{-1}f, f\rangle} = \frac{\langle Yg, g\rangle}{\|g\|^2} \in W(Y), \tag{7}$$

where $g = X_R^{-1/2} f$. The numerical range of $Y^{-1} = I + iX_R^{-1/2} X_J X_R^{-1/2}$ (and therefore its spectrum) lies on the vertical line $x = 1$. Due to the spectral mapping theorem, $\sigma(Y)$ lies on the circle $C = \{z \colon |z - 1/2| = 1/2\}$. Since Y^{-1} (and therefore Y) is normal, the numerical range $W(Y)$ is the convex hull of $\sigma(Y)$, that is, a polygon inscribed in C. In particular, $|\zeta| \le 1$ for all $\zeta \in W(Y)$. From this and (7) it follows that $|\langle X^{-1}f, f\rangle| \le \langle X_R^{-1}f, f\rangle$ for all $f \in \mathbb{C}^n$. It remains to choose $f = [1, 0 \ldots, 0]^T$. ■

Recall that the *spectral radius* $\rho(X)$ and the *numerical radius* $\omega(X)$ are defined for $X \in \mathbb{C}^{n\times n}$ as $\rho(X) = \max\{|\lambda| \colon \lambda \in \sigma(X)\}$ and $\omega(X) = \max\{|\lambda| \colon \lambda \in W(X)\}$, respectively.

It is clear that $\rho(X) \le \omega(X)$ for any matrix X, and simple examples show that the quotient $\omega(X)/\rho(X)$ can be made arbitrarily big by choosing X appropriately. However, this quotient remains bounded under certain additional conditions on X.

Lemma 2.6. *Let $X \in \mathbb{C}^{n\times n}$ be such that 0 is not an interior point of $W(X)$. Then $\omega(X)/\rho(X) \le n$.*

Proof. By scaling and rotating X, we may assume that $X_R \geq 0$ and $\rho(X) = 1$. We may also use unitary similarity to put X in upper triangular form

$$\begin{bmatrix} \lambda_1 & x_{12} & \dots & x_{1n} \\ & \ddots & & \vdots \\ & & \ddots & x_{n-1,n} \\ & & & \lambda_n \end{bmatrix}.$$

The condition $X_R \geq 0$ implies that $\begin{bmatrix} \lambda_j & x_{jk} \\ 0 & \lambda_k \end{bmatrix}_R \geq 0$ for all $j, k = 1, \dots, n$. But then $|x_{jk}| \leq \sqrt{4 \operatorname{Re} \lambda_j \operatorname{Re} \lambda_k} \leq 2\rho(X) = 2$.

It is well known that for any two matrices U and V condition $|u_{jk}| \leq v_{jk}$ $(j, k = 1, \dots, n)$ implies $\omega(U) \leq \omega(V)$ (see [2, p. 269] for the case $|u_{jk}| = v_{jk}$). Hence, $\omega(X) \leq \omega(Z)$, where Z is an upper triangular $n \times n$ matrix with

$$z_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 2 & \text{if } j < k \end{cases}. \tag{8}$$

It is also known [2, Theorem 2.1] that for any entry-wise non-negative matrix A, $\omega(A) = \rho(A_R)$. Thus $\omega(Z) = \rho(J)$, where $J = Z_R$ is the $n \times n$ matrix with all the entries equal 1. The spectrum of J consists of two eigenvalues: 0 (of multiplicity $n-1$) and a (simple) eigenvalue n, so that $\rho(J) = n$. We then see that

$$\frac{\omega(X)}{\rho(X)} = \omega(X) \leq \omega(Z) = \rho(J) = n.$$

■

Observe that the spectrum of the matrix Z is the singleton $\{1\}$ and that $W(Z)$ lies in the upper half plane. Therefore, the upper bound n for $\omega(X)/\rho(X)$ under the conditions of Lemma 2.6 is sharp.

3. Upper bound

For a given $A \in \mathbb{C}^{n \times n}$, consider its representation (3) with the biggest possible k. It is well known that the matrices A_j in such a representation are defined uniquely up to order and unitary similarities. Denote the biggest size of A_j by $u(A)$. Of course, $u(A) = 1$ if and only if A is normal; $u(A) = n$ if and only if A is unitarily irreducible.

Theorem 3.1. *For any $n \times n$ matrix A, $M(A) \leq \frac{1}{2}u(A)$.*

Proof. ¿From Lemma 2.1, it suffices to prove a (formally) weaker inequality $M(A) \leq n/2$, that is,

$$D_p(A) \leq \frac{n}{2} R_p(A)$$

for any $A \in \mathbb{C}^{n \times n}$ and an arbitrary point p located on a smooth portion of $\partial W(A)$. Considering $\tilde{A} = \alpha(A - pI)$ in place of A, we may assume that $p = 0$. Choosing an appropriate unimodular constant α, we may also assume that $W(A)$ lies in $\mathbb{C}_+$. Then from Lemma 2.2, it remains only to show that for all $n \times n$ matrices A of the form (4) with the origin located on the smooth portion of $\partial W(A)$,

$$D_0(A) \leq \frac{n}{2} R_0(A). \tag{9}$$

If the matrix A is singular, then $D_0(A) = 0$, and the claimed inequality holds trivially. Therefore, we need only consider the case where A is invertible. This implies, in particular, that $\epsilon > 0$. The numerical range A lies in $\mathbb{C}_+$ (since $A_J = 0 \oplus B_J \geq 0$) which implies $W(A^{-1}) \subset \mathbb{C}_+$. Hence, 0 is not an interior point of $W(A^{-1})$. Applying Lemma 2.6 to $X = A^{-1}$ we find that

$$D_0(A) = (\rho(A^{-1}))^{-1} \leq \frac{n}{\omega(A^{-1})}.$$

Suppose for a moment that B_J is strictly positive (and not just non-negative, as guaranteed by Lemma 2.2). Then the matrix B is invertible, and

$$(A^{-1})_{11} = \frac{\det B}{\det A} \neq 0.$$

Using an obvious inequality $|(A^{-1})_{11}| \leq \omega(A^{-1})$, we further obtain:

$$D_0(A) \leq n\frac{|\det A|}{|\det B|} = n\epsilon^2 \frac{|\det B[11]|}{|\det B|} = n\epsilon^2 |(B^{-1})_{11}|.$$

¿From this and (6) it follows that

$$\frac{D_0(A)}{R_0(A)} \leq \frac{n}{2}\frac{|(B^{-1})_{11}|}{|B_J^{-1})_{11}|} = \frac{n}{2}\frac{|(X^{-1})_{11}|}{|X_R^{-1})_{11}|},$$

where $X = -iB$. Since $X_R = B_J$, Lemma 2.5 implies the desired inequality under the additional restriction $B_J > 0$.

To remove this restriction, we reason as follows. Let A be of the form (4) with $\epsilon > 0$ and a singular non-negative B_J. Consider a family of matrices $A(\delta)$ for which B in (4) is changed to $B(\delta) = B + i\delta I$, $\delta \geq 0$. Then, of course, $B(\delta)_J = B_J + \delta I > 0$ for $\delta > 0$. Let $y = y_\delta(x)$ be the equation of $\partial W(A(\delta))$ in the neighborhood Ω of $x = 0$. Obviously, $y_\delta(0) = y'_\delta(0) = 0$, and $y''_\delta(0) = 1/R_0(A(\delta))$ (the differentiability of y_δ as a function of x for $\delta > 0$ follows from Lemma 2.3; for $\delta = 0$ we simply assume that this is the case because we are only interested in the smooth portions of $\partial W(A)$). Fix $x \in \Omega$ and $\delta > 0$. Since $x + iy_\delta(x) \in W(A(\delta))$, there exists a unit vector $z \in \mathbb{C}^n$ for which $\langle A(\delta)z, z\rangle = x + iy_\delta(x)$. But then $\mathrm{Re}\langle Az, z\rangle = x$, and $y_0(x) \leq \mathrm{Im}\langle Az, z\rangle \leq y_\delta(x)$. By Taylor's expansion,

$$0 \leq y_\delta(x) - y_0(x) = \frac{1}{2}\left(y''_\delta(\xi) - y''_0(\xi)\right) x^2$$

for some intermediate value $\xi \in (0, x)$. Dividing both sides by x^2 and taking the limit as $x \to 0$, we then see that $y''_\delta(0) \geq y''_0(0)$. Hence,

$$\frac{D_0(A)}{R_0(A)} \leq \frac{D_0(A)}{R_0(A(\delta))} = \frac{D_0(A)}{D_0(A(\delta))} \cdot \frac{D_0(A(\delta))}{R_0(A(\delta))} \leq \frac{n}{2}\frac{D_0(A)}{D_0(A(\delta))}$$

(in the last step, we use the inequality (9) for matrices $A(\delta)$ with strictly positive $B(\delta)_J$). Take the limit as $\delta \to 0$ and observe that $D_0(A(\delta)) \to D(A)$ due to the continuity of the eigenvalues as functions of the matrix's entries. ■

4. Matrices with $M(A) \leq 1$

Theorem 3.1 shows that $M(A) \leq 1$ for any matrix A with $u(A) = 2$. This, of course, also follows from Lemma 2.1 and the explicit description of $W(A)$ for 2×2 matrices A. In fact, the exact value of $M(A)$ for such matrices can be computed. For the sake of completeness, we include the result.

Theorem 4.1. *Let A be a 2×2 matrix with the eigenvalues λ_1, λ_2, and let* $s = \left(\operatorname{trace}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\right)^{1/2}$. *Then $M(A) = 0$ if $s = 0$ and*

$$M(A) = \frac{s}{\sqrt{s^2 + |\lambda_1 - \lambda_2|^2}} \tag{10}$$

otherwise.

Proof. The matrix A is normal if and only if $s = 0$; in this case $M(A) = 0$.

For $s > 0$, the matrix A is unitarily irreducible, and $W(A)$ is an ellipse with minor axis $2b = s$ and major axis $2a = \sqrt{s^2 + |\lambda_1 - \lambda_2|^2}$. The foci are, of course, located at the eigenvalues. For a current point $p \in \partial W(A)$, let x denote the distance from p to the closest eigenvalue. Then $a - c \leq x \leq a$, where $c = \sqrt{a^2 - b^2} = \frac{1}{2}|\lambda_1 - \lambda_2|$, and the distance from p to the other eigenvalue is $2a - x$. The radius of curvature at the point p is $\dfrac{(x(2a-x))^{3/2}}{ab}$ (see, for example, [13]), so that

$$M(A) = \max\{f(x) \colon a - c \leq x \leq a\},$$

where

$$f(x) = \frac{abx}{x^{3/2}(2a - x)^{3/2}} = \frac{ab}{x^{1/2}(2a - x)^{3/2}}.$$

Elementary calculus shows that $\max\{f(x) \colon a - c \leq x \leq a\} = f(a) = \dfrac{b}{a}$, which is exactly the right hand side of (10). ∎

To describe a more general situation in which $M(A) \leq 1$, recall the definition of an associated curve [8], see also [7]. For any $A \in \mathbb{C}^{n \times n}$, the equation

$$\det(uA_R + vA_J + wI) = 0,$$

with u, v, w viewed as homogeneous line coordinates, defines an algebraic curve of class n. The real part of this curve, denoted by $C(A)$, is the *associated curve* of A. The n real foci of $C(A)$ are the eigenvalues of A, and the convex hull of $C(A)$ coincides with $W(A)$.

Theorem 4.2. *Let $A \in \mathbb{C}^{n \times n}$ be such that its associated curve consists only of points and ellipses. Then $M(A) \leq 1$.*

Proof. Any point p located on the smooth portion of $\partial W(A)$ lies on one of the ellipses E constituting $C(A)$. Hence, the distance from p to one of the foci of E does not exceed $R_p(A)$. It remains to recall that the foci of E are at the same time foci of $C(A)$, that is, belong to $\sigma(A)$. ∎

It is interesting to observe that there exist matrices A with $u(A) > 2$ satisfying Theorem 4.2. An example of a unitarily irreducible 4×4 matrix A where $C(A)$ is a union of two circles (once circle does not contain the other) was given in [9]. From [10], all $(0, 1)$-matrices with at most one 1 in each row and column have $C(A)$ consisting of points and concentric circles, and therefore also satisfy Theorem 4.2.

5. Lower bound

In this section, we consider an alternative approach to computing the quotient $D_p(A)/R_p(A)$, which leads to some lower bounds for M_n. For any $A \in \mathbb{C}^{n \times n}$, let $\lambda(\theta)$ denote the maximum eigenvalue of $A_R \cos\theta + A_J \sin\theta$. It is well known that

λ is an analytic function of θ (possibly except for some isolated points), and that $\partial W(A)$ admits a parametric representation

$$\begin{aligned} x(\theta) &= \lambda(\theta)\cos\theta - \lambda'(\theta)\sin\theta, \\ y(\theta) &= \lambda(\theta)\sin\theta + \lambda'(\theta)\cos\theta \end{aligned} \tag{11}$$

(again, with possible exception of finitely many points). The radius of curvature of $\partial W(A)$ at $p = (x(\theta), y(\theta))$ equals

$$R(\theta) = \lambda''(\theta) + \lambda(\theta) \tag{12}$$

(see, i.e., [11], where formulas (11) and (12) are mentioned explicitly).

¿From Section 3, it seems natural to consider matrices of the form $A = Z^{-1}$, where Z is an $n \times n$ triangular matrix given by (8), as possible candidiates for producing large $D_p(A)/R_p(A)$. A direct computation shows that $Z^{-1} = VZV$, where $V = \operatorname{diag}[1, -1, \dots, (-1)^n]$. Hence, Z^{-1} is unitarily similar to Z, and we let $A = Z$. Then

$$(A_R\cos\theta + A_J\sin\theta - \lambda I)_{jk} = \begin{cases} \cos\theta - \lambda & \text{if } j = k \\ \cos\theta - i\sin\theta & \text{if } j < k \\ \cos\theta + i\sin\theta & \text{if } j > k \end{cases}.$$

¿From [12, Problem 392] it follows that

$$\det(A_R\cos\theta + A_J\sin\theta - \lambda I) = \\ (-1)^n \frac{(\cos\theta - i\sin\theta)(\lambda + i\sin\theta)^n - (\cos\theta + i\sin\theta)(\lambda - i\sin\theta)^n}{2i\sin\theta}.$$

Hence,

$$\lambda(\theta) = \sin\theta\cot\frac{\theta}{n}, \qquad \theta \in [-\pi, \pi]$$

with $\lambda(0) = n$ defined by continuity. Consequently,

$$\lambda'(\theta) = \cos\theta\cot\frac{\theta}{n} - \frac{1}{n}\sin\theta\csc^2\frac{\theta}{n},$$

and

$$\lambda''(\theta) = -\sin\theta\cot\frac{\theta}{n} - \frac{2}{n}\cos\theta\csc^2\frac{\theta}{n} + \frac{2}{n^2}\cos\frac{\theta}{n}\sin\theta\csc^3\frac{\theta}{n}.$$

Formulas (11) and (12) yield

$$x(\theta) = \frac{1}{n}\sin^2\theta\csc^2\frac{\theta}{n}, \qquad y(\theta) = \cot\frac{\theta}{n} - \frac{1}{n}\sin\theta\cos\theta\csc^2\frac{\theta}{n} \tag{13}$$

and

$$R(\theta) = \frac{2}{n^2}\left(\sin\theta\cos\frac{\theta}{n} - n\cos\theta\sin\frac{\theta}{n}\right)\csc^3\frac{\theta}{n}, \tag{14}$$

respectively.

The value $\theta = \pi$ corresponds to the point $i\cot\frac{\pi}{n}$ located at the "flattening" of $\partial W(A)$. The distance from this point to the (only) eigenvalue 1 of A is $D(\pi) = \csc\frac{\pi}{n}$, while $R(\pi) = \frac{2}{n}\csc^2\frac{\pi}{n}$. Hence, $D(\pi)/R(\pi) = \frac{n}{2}\sin\frac{\pi}{n}$, which leads to the following

Theorem 5.1. $M_n \geq \dfrac{n}{2}\sin\dfrac{\pi}{n}$.

When $\theta \to 0$ in formulas (13), (14), we see that $x(0) = n$, $y(0) = 0$, $R(0) = \frac{2(n^2-1)}{3n}$. So,

$$\frac{D(0)}{R(0)} = \frac{3n(n-1)}{2(n^2-1)} = \frac{3n}{2(n+1)}.$$

For $n = 2$, this quotient is the same as $D(\pi)/R(\pi) = 1$. This is not surprising: the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has a circular numerical range $W(A)$, so that $D(\theta) \equiv R(\theta)$ (= 1). Of course, formulas (13) and (14) give the same conclusion.

For $n \geq 3$, however,

$$\frac{3n}{2(n+1)} < \frac{n}{2} \sin \frac{\pi}{n}.$$

We suspect that for matrices under consideration, $\sup_\theta D(\theta)/R(\theta)$ is assumed at $\theta = \pi$. The next statement confirms this conjecture for $n = 3$.

Theorem 5.2. *Let*

$$A = \begin{bmatrix} \lambda & x & y \\ 0 & \lambda & z \\ 0 & 0 & \lambda \end{bmatrix} \tag{15}$$

with $|x| = |y| = |z| \neq 0$. *Then* $M(A) = \dfrac{3\sqrt{3}}{4}$.

Proof. As was shown in [7], the associated curve $C(A)$ for the matrix (15) is a cardioid. By scaling, rotating and shifting A we may without loss of generality suppose that this cardioid is given by the polar equation

$$r = \frac{2}{3}(1 + \cos\theta), \qquad -\pi \leq \theta \leq \pi.$$

The numerical range $W(A)$ then coincides with the convex hull of the portion of $C(A)$ corresponding to $\theta \in [-2\pi/3, 2\pi/3]$, and the triple eigenvalue of A is $\lambda = 1/3$. Direct computations show that, for a point $p = (x, y)$ on the non-flat portion of $\partial W(A)$:

$$D_p(A) = \sqrt{(x - \frac{1}{3})^2 + y^2} = \sqrt{r^2 - \frac{2}{3} r \cos\theta + \frac{1}{9}} = \frac{1}{3}\sqrt{5 + 4\cos\theta},$$

$$R_p(A) = \frac{(r^2 + (r')^2)^{3/2}}{r^2 + 2(r')^2 - rr''} = \frac{4\sqrt{2}}{9}(1 + \cos\theta)^{1/2}.$$

Hence,

$$\frac{D_p(A)}{R_p(A)} = \frac{3}{4\sqrt{2}} \sqrt{4 + \frac{1}{1 + \cos\theta}},$$

and

$$M(A) = \frac{3}{4\sqrt{2}} \max_{0 \leq \theta \leq 2\pi/3} \sqrt{4 + \frac{1}{1 + \cos\theta}} = \frac{3}{4\sqrt{2}} \sqrt{4 + \frac{1}{1 + \cos\frac{2\pi}{3}}} = \frac{3\sqrt{3}}{4}.$$

■

According to [8], there are three possible shapes of $W(A)$ for unitarily irreducible 3×3 matrices: an ellipse, an ovular shape, and a shape with a flat portion on the boundary. Of course, $M(A) \leq 1$ for all matrices with an elliptical $W(A)$. As it happens [7], all 3×3 matrices with a flat portion on $\partial W(A)$ and coinciding eigenvalues are unitarily similar to a matrix (15). Hence, for all such matrices $M(A) = 3\sqrt{3}/4$. We did not compute the explicit values of $M(A)$ for 3×3 matrices A with ovular $W(A)$.

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