Linear Preservers of Isomorphic Lattices of Invariant Operator Ranges

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Abstract

We describe all linear self-mappings of the space of bounded linear operators in an infinite dimensional separable complex Hilbert space which preserve the isomorphism class of the lattice of invariant operator ranges.

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1 Main Results

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. Let $\mathcal{L}(\mathcal{H})$ denote the Banach algebra of linear bounded operators on $\mathcal{H}$ with the operator norm. An *operator range* is, by definition, a linear set $\mathcal{M} \subseteq \mathcal{H}$ such that

$$\mathcal{M} = \operatorname{Range} G := \{Gx | x \in \mathcal{H}\}$$

for some $G \in \mathcal{L}(\mathcal{H})$. Equivalently, $\mathcal{M} \subseteq \mathcal{H}$ is an operator range if and only if $\mathcal{M} = \operatorname{Range} G$ for some linear bounded operator $G : \mathcal{H}_0 \to \mathcal{H}$ with zero kernel, where $\mathcal{H}_0$ is a suitable Hilbert space.

If $T \in \mathcal{L}(\mathcal{H})$, we denote by $\mathcal{IOR}(T)$ the set of all operator ranges $\mathcal{M}$ that are T-invariant: $Tx \in \mathcal{M}$ for every $x \in \mathcal{M}$. The set $\mathcal{IOR}(T)$ is a lattice (with respect to addition and intersection); this follows from the general fact that intersection and sum of two operator ranges are again operator ranges. For a proof of this fact and for other fundamental properties of operator ranges see, for example, [2].

In this paper we prove two theorems:

Theorem 1 *For every $T \in \mathcal{L}(\mathcal{H})$, if $\mathcal{M}_1 \subset \mathcal{M}_2$ are two T-invariant operator ranges such that the dimension of the factor linear set $\mathcal{M}_2/\mathcal{M}_1$ exceeds one, then there exists $\mathcal{M} \in \mathcal{IOR}(T)$ with the property that*

$$\mathcal{M}_1 \subset \mathcal{M} \subset \mathcal{M}_2, \qquad \mathcal{M}_1 \neq \mathcal{M} \neq \mathcal{M}_2.$$

Theorem 2 *Let $\phi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})$ be a bijective linear map such that for every $T \in \mathcal{L}(\mathcal{H})$, the lattices $\mathcal{IOR}(T)$ and $\mathcal{IOR}(\phi(T))$ are isomorphic. Then there exists a non-zero complex number α, a boundedly invertible $S \in \mathcal{L}(\mathcal{H})$, and a (not necessarily continuous) linear functional $f : \mathcal{L}(\mathcal{H}) \longrightarrow \mathbb{C}$ such that*

$$\phi(T) = \alpha STS^{-1} + f(T)I \tag{1}$$

for every $T \in \mathcal{L}(\mathcal{H})$.

It was proved in [3] that the same formula (1) describes the bijective linear maps ϕ on $\mathcal{L}(\mathcal{H})$ with the property that the lattice of T-invariant linear sets and the lattice of $\phi(T)$-invariant linear sets are isomorphic, for every $T \in \mathcal{L}(\mathcal{H})$. Combining this result with Theorem 2, we obtain:

Corollary 3 *A bijective linear map $\phi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})$ has the property that for every $T \in \mathcal{L}(\mathcal{H})$, the lattices $\mathcal{IOR}(T)$ and $\mathcal{IOR}(\phi(T))$ are isomorphic, if and only if ϕ has the property that for every $T \in \mathcal{L}(\mathcal{H})$, the lattices of T-invariant linear sets and of $\phi(T)$-invariant linear sets are isomorphic.*

Theorem 1 will be used in the proof of Theorem 2. Perhaps, Theorem 1 is independently interesting.

2 Proof of Theorem 1

We start with some preliminaries. Let $\mathcal{N}$ be an operator range. There is a norm $\|\cdot\|_{\mathcal{N}}$ on $\mathcal{N}$ with respect to which $\mathcal{N}$ is a Hilbert space, and in addition,

$$\|x\|_{\mathcal{N}} \geq \|x\|_{\mathcal{H}} \tag{2}$$

for every $x \in \mathcal{N}$, where $\|\cdot\|_{\mathcal{H}}$ is the norm in $\mathcal{H}$ (see Theorem 1.1 of [2]). In fact, if $\mathcal{N} = \operatorname{Range} G$, where $G : \mathcal{H}_0 \to \mathcal{H}$ is a linear bounded operator with zero kernel, then one can choose $\|\cdot\|_{\mathcal{N}}$ so that

$$\|Gy\|_{\mathcal{N}}^2 = \|Gy\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}_0}^2, \quad y \in \mathcal{H}_0. \tag{3}$$

Lemma 4 *If $T \in \mathcal{L}(\mathcal{H})$, and if $\mathcal{N}$ is a T-invariant operator range, then T is bounded, as an operator on the Hilbert space $\mathcal{N}$.*

Proof. By the closed graph theorem, we only have to check that the graph of T is closed in the Hilbert space $\mathcal{N} \oplus \mathcal{N}$. Let a sequence $\{(x_n, Tx_n) \in \mathcal{N} \oplus \mathcal{N}\}_{n=1}^{\infty}$ converge to $(y, z) \in \mathcal{N} \oplus \mathcal{N}$. Then $x_n \to y$ and $Tx_n \to z$ in $\mathcal{N}$, therefore also $x_n \to y$ and $Tx_n \to z$ in $\mathcal{H}$. Since $T \in \mathcal{L}(\mathcal{H})$, we must have $z = Ty$, which proves the closedness of the graph of T in $\mathcal{N} \oplus \mathcal{N}$. □

Lemma 5 *The set of operator ranges in the Hilbert space $\mathcal{N}$ (in short: $\mathcal{N}$-operator ranges) coincides with the set of operator ranges in the Hilbert space $\mathcal{H}$ (in short: $\mathcal{H}$-operator ranges), that are contained in $\mathcal{N}$.*

Proof. Let $G : \mathcal{H}_0 \to \mathcal{H}$ be a linear bounded operator with zero kernel and range $\mathcal{N}$, and assume that $\|\cdot\|_{\mathcal{N}}$ is given by (3). If $\operatorname{Range} B \subseteq \mathcal{N}$ for some $B \in \mathcal{L}(\mathcal{H})$, then by Douglas' lemma, there exists $C \in \mathcal{L}(\mathcal{H}, \mathcal{H}_0)$ such that $B = GC$. Therefore,

$$\|By\|_{\mathcal{N}}^2 = \|GCy\|_{\mathcal{N}}^2 = \|By\|_{\mathcal{H}}^2 + \|Cy\|_{\mathcal{H}_0}^2 \leq (\|B\|^2 + \|C\|^2)\|y\|_{\mathcal{H}}^2,$$

and so B is a bounded operator from $\mathcal{H}$ into $\mathcal{N}$. Hence $\operatorname{Range} B$ is an $\mathcal{N}$-operator range. Conversely, if $\mathcal{M} = \operatorname{Range} B$, $B \in \mathcal{L}(\mathcal{N})$ is an $\mathcal{N}$-operator range, then (2) shows that B is bounded as an operator into $\mathcal{H}$, and so $\mathcal{M}$ is an $\mathcal{H}$-operator range. □

Proof of Theorem 1. Let $T \in \mathcal{L}(\mathcal{H})$, and fix two T-invariant operator ranges $\mathcal{M}_1 \subset \mathcal{M}_2$ satisfying the hypotheses of Theorem 1. In view of Lemmas 4 and 5 (applied for $\mathcal{N} = \mathcal{M}_2$), we can (and do) assume that $\mathcal{M}_2 = \mathcal{H}$.

Let us consider three possibilities:

(i) $\mathcal{M}_1$ is not closed and not dense in $\mathcal{H}$. We are done - take $\mathcal{M}$ to be the closure of $\mathcal{M}_1$.

(ii) $\mathcal{M}_1$ is closed. Note that every $\hat{T} \in \mathcal{L}(\mathcal{H}_0)$, where the dimension of the Hilbert space $\mathcal{H}_0$ exceeds one, has an invariant operator range different from $\{0\}$ and $\mathcal{H}_0$. Indeed, leaving aside the trivial case of a scalar operator $\hat{T}$, since the spectrum of $\hat{T}$ is not empty, for some $\lambda \in \mathbb{C}$ we will have $\operatorname{Ker}(\hat{T} - \lambda I) \neq \{0\}$ or $\operatorname{Range}(\hat{T} - \lambda I) \neq \mathcal{H}_0$. So we may take $\operatorname{Ker}(\hat{T} - \lambda I)$ or $\operatorname{Range}(\hat{T} - \lambda I)$, as appropriate, as the required operator range. Applying the observation to the operator $\hat{T}$ induced by T in the factor space $\mathcal{H}/\mathcal{M}_1$, we complete the proof of Theorem 1 in case $\mathcal{M}_1$ is closed.

(iii) $\mathcal{M}_1$ is dense in $\mathcal{H}$. We have $\mathcal{M}_1 = \operatorname{Range} V$, where V is a bounded positive operator on $\mathcal{H}$ (see [2]). Moreover, by Lemma 4, T is bounded as an operator on the Hilbert space $\mathcal{M}_1$. It is also bounded as an operator on the Hilbert space $\mathcal{H}$. Therefore, by Donoghue's Theorem [1], the operator T maps $\operatorname{Range} \phi(V)$ into itself for every Löwner function ϕ (in fact, it is sufficient to use a much easier result with $\phi(t) = t^\alpha$, $0 < \alpha < 1$, see, e.g., [4], Theorem 4.1.10). Using a description of $\operatorname{Range} V^\alpha, 0 < \alpha < 1$, in terms of the spectral decomposition of V, one can easily check that these operator ranges are properly contained in $\mathcal{H}$ and properly contain $\mathcal{M}_1$. Thus, we obtain a continuum of required T-invariant operator ranges.

3 Proof of Theorem 2

The proof follows the pattern of the proof of Theorem 3.1 in [3]. We need several lemmas, in analogy with the proof given in [3]. In what follows, we denote by lat_n (resp. $\operatorname{lat}_\infty$) the lattice of operator ranges in the n-dimensional ($n < \infty$) (resp. infinite dimensional separable) Hilbert space.

We start with a known result on operator ranges.

Lemma 6 *Let $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{M} \neq \mathcal{H}$ is an operator range in $\mathcal{H}$, then there exists a nonzero operator range $\mathcal{N}$ in $\mathcal{H}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$.*

Proof. The statement is clear if $\mathcal{M}$ is closed. Otherwise, by a result of von Neumann (see [2] for a transparent proof due to Dixmier) there exists a unitary operator U such that $\mathcal{M} \cap U\mathcal{M} = \{0\}$, so we may take $\mathcal{N} = U\mathcal{M}$. □

Lemma 7 *Let $\mathcal{H}$ be a separable Hilbert space, and let $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{IOR}(T)$ is isomorphic as a lattice to lat_n, $n < \infty$ (resp. $\operatorname{lat}_\infty$). Then T is a scalar multiple of the identity and $\dim \mathcal{H} = n$ (resp. $\mathcal{H}$ is infinite dimensional).*

Proof. Assume first that $\mathcal{IOR}(T)$ is isomorphic to lat_n, $n < \infty$. Then every chain

$$\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots \subseteq \mathcal{M}_m, \quad \mathcal{M}_j \in \mathcal{IOR}(T), \quad j = 1, 2, \cdots, m$$

has at most $n+1$ distinct elements, and there exists such a chain with exactly $n+1$ distinst elements. By Theorem 1, $\dim \mathcal{H} = n$. Proposition 2.5 of [3] shows that T has the required form.

Now assume that $\mathcal{IOR}(T)$ is isomorphic to lat_∞. Since every nonzero element of lat_∞ contains a minimal nonzero element, namely, a one-dimensional subspace, the same is true of $\mathcal{IOR}(T)$. By Theorem 1, a minimal nonzero element of $\mathcal{IOR}(T)$ must be a one-dimensional subspace, i.e., the subspace spanned by an eigenvector of T. We obtain that every nonzero T-invariant operator range contains an eigenvector.

Let $\tau : \mathcal{IOR}(T) \to \text{lat}_\infty$ be an isomorphism, where lat_∞ is the lattice of operator ranges in an infinite dimensional separable Hilbert space $\mathcal{H}_0$. Assume that u and v are linearly independent eigenvectors of T corresponding to eigenvalues λ and μ, respectively. The subspace

$$\tau\left((\text{span}\, u) + (\text{span}\, v)\right) \subset \mathcal{H}_0$$

is clearly two-dimensional, and therefore contains infinitely many different elements of lat_∞. So the element

$$(\text{span}\, u) + (\text{span}\, v) \in \mathcal{IOR}(T) \tag{4}$$

also contains infinitely many different elements of $\mathcal{IOR}(T)$. However, (4) contains infinitely many T-invariant subspaces if and only if $\lambda = \mu$. We obtain that T has only one eigenvalue (perhaps of high multiplicity), call it λ_0.

If $\text{Ker}\,(T - \lambda_0 I) \neq \mathcal{H}$, then $\tau\left(\text{Ker}\,(T - \lambda_0 I)\right) \neq \mathcal{H}_0$. By Lemma 6, there exists $\mathcal{M} \in \text{lat}_\infty$, $\mathcal{M} \neq \{0\}$, such that

$$\tau\left(\text{Ker}\,(T - \lambda_0 I)\right) \cap \mathcal{M} = \{0\}.$$

Then $\tau^{-1}(\mathcal{M})$ is a nonzero T-invariant operator range that has the zero intersection with $\text{Ker}\,(T - \lambda_0 I)$. On the other hand, we have seen above that $\tau^{-1}(\mathcal{M})$ contains an eigenvector of T corresponding to the eigenvalue λ_0, a contradiction. So we must conclude that $\text{Ker}\,(T - \lambda_0 I) = \mathcal{H}$. □

Lemma 8 *Let $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional separable Hilbert space. Then the following are equivalent:*

(a) $T = \alpha P + \beta I$ with $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$, $P = P^2$, and $\text{rank}\, P = n < \infty$;

(b) $\mathcal{IOR}(T)$ is isomorphic as a lattice to $\text{lat}_n \oplus \text{lat}_\infty$.

Proof. Assume (a) holds. Clearly, $\mathcal{IOR}(T) = \mathcal{IOR}(P)$. Since every P-invariant operator range $\mathcal{M}$ is of the form $\mathcal{M} = P\mathcal{M} + (I - P)\mathcal{M}$, it follows that $\mathcal{IOR}(P)$ is isomorphic to

$$(P\text{lat}_\infty) \oplus ((I - P)\text{lat}_\infty),$$

where we identify lat_∞ with the lattice of operator ranges in $\mathcal{H}$. By Lemma 5, $(I-P)\mathrm{lat}_\infty$ coincides with the lattice of operator ranges in $\operatorname{Ker} P$, which in turn is isomorphic to lat_∞. Thus (b) holds.

Conversely, assume (b) holds. Fix a lattice isomorphism $\tau : \mathcal{IOR}(T) \to \mathrm{lat}_n \oplus \mathrm{lat}_\infty$. Let $\mathcal{M}_1 = \tau^{-1}(\mathbb{C}^n \oplus \{0\})$ and $\mathcal{M}_2 = \tau^{-1}(\{0\} \oplus \mathcal{H}_0)$. Consider $\mathcal{M}_2$ as a Hilbert space, and T as a linear bounded operator on $\mathcal{M}_2$ (see Lemma 4). Taking into account that the lattice of $T|_{\mathcal{M}_2}$-invariant $\mathcal{M}_2$-operator ranges coincides with the sublattice of those T-invariant $\mathcal{H}$-operator ranges that are contained in $\mathcal{M}_2$ (see Lemma 5), we obtain from Lemma 7 that $T|_{\mathcal{M}_2} = \gamma I$ for some $\gamma \in \mathbb{C}$. Analogously, $T|_{\mathcal{M}_1} = \delta I$ for some $\delta \in \mathbb{C}$.

It turns out that $\gamma \neq \delta$. Indeed, arguing by contradiction, assume that T is a scalar operator. Let $\mathcal{N} \in \mathcal{IOR}(T)$ be any element with the property that every chain

$$\{0\} = \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots \subseteq \mathcal{N}_{m-1} \subseteq \mathcal{N}_m = \mathcal{M}, \quad \{0\} \neq \mathcal{N}_2 \neq \cdots \neq \mathcal{N}_{m-1} \neq \mathcal{M},\ \mathcal{N}_j \in \mathcal{IOR}(T) \tag{5}$$

has length 3 (i.e., $m = 3$), in other words, $\dim \mathcal{N} = 2$. Then obviously there exists a continuum of $\mathcal{N}_2 \in \mathcal{IOR}(T)$ that satisfy (5). However, the element $\mathcal{N} = \mathcal{V} \oplus \mathcal{U} \in \mathrm{lat}_n \oplus \mathrm{lat}_\infty$, where $\mathcal{V}$ and $\mathcal{U}$ are one-dimensional subspaces of $\mathbb{C}^n$ and of $\mathcal{H}_0$, respectively, has the property that every chain

$$\{0\} = \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots \subseteq \mathcal{N}_{m-1} \subseteq \mathcal{N}_m = \mathcal{N}, \quad \{0\} \neq \mathcal{N}_2 \neq \cdots \neq \mathcal{N}_{m-1} \neq \mathcal{N},\ \mathcal{N}_j \in \mathrm{lat}_n \oplus \mathrm{lat}_\infty \tag{6}$$

has length 3, but there exist only two elements $\mathcal{N}_2$ that satisfy (6). This contradicts the hypothesis (b).

Once we have ascertained that $\gamma \neq \delta$, (a) follows with $\alpha = \delta - \gamma$, and with P the projection on $\mathcal{M}_1$ along $\mathcal{M}_2$. ☐

If P is assumed to have infinite dimensional rank and kernel, then the analogue of Lemma 8 runs as follows, with essentially the same proof as Lemma 8:

Lemma 9 *Let T be as in Lemma 8. Then the following are equivalent:*

(a) $T = \alpha P + \beta I$ with $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$, $P = P^2$, and $\dim \operatorname{Range} P = \dim \operatorname{Ker} P = \infty$;

(b) $\mathcal{IOR}(T)$ is isomorphic as a lattice to $\mathrm{lat}_\infty \oplus \mathrm{lat}_\infty$.

Lemma 10 *Let $E = \{e_j\}_{j=1}^\infty$ be an orthonormal basis in $\mathcal{H}$. Then there exists $T \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{IOR}(T)$ is not isomorphic to $\mathcal{IOR}(T^t)$, where $T^t \in \mathcal{L}(\mathcal{H})$ is the operator whose infinite matrix with respect to the basis E is the transpose of the infinite matrix representing T (with respect to E).*

Proof. Define T by $Te_j = e_{j+1}$, $j = 1, 2, \cdots$. Clearly, $T^t e_j = e_{j-1}$ for $j = 2, 3, \cdots$, and $T^t e_1 = 0$. The linear span of e_1 is a minimal nonzero element of $\mathcal{IOR}(T^t)$. If $\mathcal{IOR}(T)$ and $\mathcal{IOR}(T^t)$ were isomorphic, then $\mathcal{IOR}(T)$ would also have a minimal nonzero element, which by Theorem 1 would have to be a one-dimensional subspace. However, this is impossible, because $\mathrm{Ker}\,(\lambda I - T) = \{0\}$ for every $\lambda \in \mathbb{C}$. □

Once Lemmas 7 - 10 are established, the proof of Theorem 2 proceeds as that of Theorem 3.1 in [3].

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