# CONVEX HULLS OF COXETER GROUPS

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Abstract. We survey known and new results concerning the geometric structure of the convex hulls of finite irreducible Coxeter groups. In particular we consider a conjecture concerning the normals to the faces of maximal dimension of these convex hulls. This conjecture is related to a theorem of Birkhoff and also to interpolation of operators. We describe various approaches to its proof as well as various computer calculations involved.

## 1. Introduction

Let $G$ be a finite irreducible Coxeter group naturally acting on a finite dimensional real Euclidean space $V$. So, $G$ is a subset of the linear space End $V$ of linear operators in $V$. Let $\mathbf{I}$ denote the identity operator.

We study the geometry of the convex hull of $G$, which we denote by $\operatorname{conv} G$. This is a convex polytope in the linear space End $V$. What is its facial structure? In particular, what are its faces of maximal dimension? All these problems naturally arise in various disguises — we were mostly motivated by a duality approach to interpolation of operators discussed below.

Recently there has been substantial progress in this direction, see [10]. During the summer of 2000 we were able to move further, relying heavily on computer calculations. The goal of this article is to give a full account of the present state of the problem.

### 1.1. Birkhoff's Theorem.

To describe the results, we start with a formulation of a well known theorem due to G. Birkhoff [2].

**Definition 1.1.** *Let $T = (t_{ij})$ be an $n \times n$ matrix. It is called* **bistochastic** *(or* **doubly stochastic***) if its entries are non-negative and*

$$\text{for every } j,\ 1 \le j \le n, \qquad \sum_{i=1}^{n} t_{ij} = 1, \qquad \sum_{i=1}^{n} t_{ji} = 1$$

*The set of all bistochastic $n \times n$ matrices is denoted by $\Omega_n$.*

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Obviously, the set $\Omega_n$ is a convex polytope in the space of $n \times n$ matrices. Since there are $2n-1$ independent linear equations involved in the definition, this polytope is actually in a subspace of dimension $n^2 - (2n-1) = (n-1)^2$.

**Definition 1.2.** *Let* $\mathrm{Perm}_n$ *denote the group of* $n \times n$ *permutation matrices.*

**Theorem 1.3** (G. Birkhoff, [2])**.** *The set of vertices of the polytope* $\Omega_n$ *coincides with the set of permutation matrices* $\mathrm{Perm}_n$*.*

So, the polytope of bistochastic matrices turns out to be nothing else but the convex hull of the group $\mathrm{Perm}_n$ . Let us reformulate this result. The group $\mathrm{Perm}_n$ acts reducibly on $\mathbf{R}^n$ : it fixes the vector $e = (1, 1, \cdots, 1)$ and acts irreducibly on its orthogonal complement $e^\perp = \{(x_i) \in \mathbf{R}^n : \sum x_i = 0\}$. Consider the group $A_{n-1} = \mathrm{Perm}_n |_{e^\perp}$. This group is maybe the most important example of a finite irreducible Coxeter group. It spans a $(n-1)^2$-dimensional subspace in the space of $n \times n$ matrices. Note that all bistochastic matrices also fix the vector $e$ and leave $e^\perp$ invariant — this is actually a part of the definition. So, Theorem 1.3 says that the polytope of bistochastic matrices is located in a $(n-1)^2$-dimensional subspace and its vertices are operators from $A_{n-1}$. Thus the definition of bistochastic matrices describes the convex hull of the Coxeter group $A_{n-1}$ in terms of linear inequalities. One can rather easily see that this set of inequalities is the smallest possible: each inequality describes a half-space bounded by a face of the polytope. But what is the invariant meaning of these inequalities?

There exists a whole industry dealing with generalizations of results known for groups $A_n$ to other Coxeter groups. Very often these generalizations turn out to be non-trivial and useful, providing deeper insights into the results. Sometimes they are simply exercises in the theory of Coxeter groups. It is usually very interesting if a result valid for $A_n$ proves not to be valid for all Coxeter groups — such results are usually most challenging. This is exactly the case with generalizations of the Birkhoff Theorem: a natural generalization — see Conjecture 1.4 below — is not true for Coxeter groups whose graphs are branching, but it seems to be true for Coxeter groups whose graphs are non-branching (as of January 2001, Conjecture 1.4 has been verified for all finite irreducible Coxeter groups except $H_4$).

Actually, our main impetus came not from a simple (though natural) desire to generalize but from a rather unexpected source — the theory of Interpolation of Operators. The problem becomes very natural in that setting and the main Conjecture arises from some deep results related to interpolation of operators in spaces with Coxeter-invariant norms. We describe this circle of ideas in Subsection 1.3 below.

1.2. **Conjecture.** Returning to the general setting of an irreducible Coxeter group $G$ we may say that we are interested in calculation of the faces of $\mathrm{conv}\, G$ of maximal dimension $(\dim V)^2 - 1$. Each such face is a polytope of full dimension in an affine hyperplane $\{A \in \mathrm{End}\ V : f(A) = c\}$ where $f$ is a linear functional on End $V$. These (properly scaled) functionals are naturally identified with elements of $\mathrm{Extr}\,(\mathrm{conv}\, G)^\circ$ — the set of extreme elements of the polar polytope

$$(\mathrm{conv}\, G)^\circ = \{h \in (\mathrm{End}\ V)^* : \forall\, A \in \mathrm{conv}\, G\ h(A) \le 1\}.$$

We prefer to introduce a Euclidean structure into End $V$ which allows to identify the spaces End $V$ and $(\mathrm{End}\ V)^*$ and to treat the mentioned functionals as normals to the face. The needed scalar product on End $V$ is given by the formula $(A, B) =$

trace $(AB^*)$, where $B^*$ is the operator adjoint to $B$ (to define $B^*$ we need the Euclidean structure in $V$). Moreover, the Euclidean structure in $V$ allows to identify $V^*$ with $V$ and therefore to identify $V \otimes V$ with $V \otimes V^*$. In turn, $V \otimes V^*$ is naturally identified with the space $(\text{End } V)^*$, which is already identified with End $V$, so we may identify the spaces $V \otimes V$ and End $V$. In particular, for any $x, y \in V$ we identify $x \otimes y$ with the rank one operator $z \mapsto x\langle y, z\rangle$.

Let us describe the Conjecture.

Consider the set $\mathcal{W}(G)$ of all weights of the group $G$. Throughout this paper we consider "non-normalized weights", i.e., nonzero vectors directed along extreme rays of Weyl chambers – the term "weight" is often used to denote only specially normalized vectors of this type. Each weight $\omega$ is associated with a vertex $\pi(\omega)$ of the Coxeter graph $\Gamma(G)$.

Let $E_G$ denote the set of **extremal weights** of the group $G$, i.e., those associated with the **end vertices** of the Coxeter graph $\Gamma(G)$.

Put $m_G(x, y) = \max\{\langle gx, y\rangle : g \in G\}$. One can show that in the case of an irreducible Coxeter group $G$ the quantity $m_G(x, y)$ is strictly positive for any nonzero vectors $x, y \in V$.

Let

$$\mathcal{B}_G = \{\omega \otimes \tau / m_G(\omega, \tau) : \omega, \tau \in E_G, \pi(\omega) \neq \pi(\tau)\}.$$

We call the elements of $\mathcal{B}_G$ the **Birkhoff tensors.** Note that Birkhoff tensors all have rank one.

The importance of Birkhoff tensors for our problem is apparent because of the following result (see Theorem 4.4 below):

$$\mathcal{B}_G = (\text{Extr}\,(\text{conv}\,G)^\circ) \bigcap (\text{ rank 1 tensors }).$$

The following conjecture was first proposed in 1979 by Veronica Zobin [19] and later elaborated by the last author:

**Conjecture 1.4.** *(a) If the Coxeter graph $\Gamma(G)$ is non-branching then*

$$\mathcal{B}_G = \text{Extr}\,(\text{conv}\,G)^\circ.$$

*(b) If the Coxeter graph $\Gamma(G)$ is branching then*

$$\mathcal{B}_G \subsetneqq \text{Extr}\,(\text{conv}\,G)^\circ.$$

Part (b) of the Conjecture was proved in [10]; we reproduce this proof in Section 8 below. As for Part (a), it was proved in [10] for all infinite families of Coxeter groups with non-branching graphs, and we have verified it for the groups $F_4$ and $H_3$ by rather nontrivial computer calculations. The only remaining group is $H_4$. The computer calculations that were successful for other groups could not be completed for $H_4$ on the available computers, mainly because of insufficient random access memory.

It should be noted that the success in proving Part (b) was achieved with a very strong computer component: a computer calculation found an essentially unique tensor of rank 3 belonging to the set Extr (conv $D_4)^\circ$, and then the general case of a Coxeter group with a branching graph was reduced to this one. We still do not quite understand the invariant meaning of this rank 3 tensor. However, after this tensor is found one can verify by hand that it really belongs to Extr (conv $D_4)^\circ$, so the proof does not formally depend upon the computer calculations.

Certainly, it would be very interesting to find a unified approach avoiding the case study of irreducible Coxeter groups and heavy use of computers. We believe that there must be a general simple reason for the validity of the Conjecture.

1.3. **Interpolation of operators.** The above conjecture naturally appeared in the theory of interpolation of operators on spaces with given symmetries — see [16, 15]. The main object studied in these papers was the convex set env $G$ defined as the semigroup of all linear operators in $V$ which transform every $G$-invariant convex closed set into itself:

$$\text{env } G = \{T \in \text{End } V : T(U) \subset U \text{ for every convex closed } G\text{-invariant } U \subset V\}.$$

Obviously,

$$\operatorname{conv} G \subset \text{env } G.$$

If these two sets coincide then the only operators simultaneously contracting all $G$-invariant closed convex sets are those which have this property almost by definition. So this case is not interesting from the point of view of interpolation of operators. The opposite case is much more interesting — there are nontrivial operators that can be interpolated. In the case of an irreducible finite Coxeter group $G$ we have a convenient dual description of the set env $G$ – this description is one of the central results of [16]:

$$\operatorname{Extr}(\text{env } G)^\circ = \mathcal{B}_G.$$

So the question is if $\operatorname{conv} G = \text{env } G$, i.e., if $\operatorname{Extr}(G^\circ) = \mathcal{B}_G$.

Currently we know that the latter equality is not true for Coxeter groups with branching graphs. This leads to two difficult problems. First, what is $\operatorname{Extr}(G^\circ)$ for such groups? Second, what are the extreme common contractions, i.e., the extreme elements of the semigroup env $G$? As of now, we have no viable conjectures.

1.4. **Geometry of orbihedra.** The problem of describing the facial structure of $\operatorname{conv} G$ is a particular case of a more general problem, which naturally arises in several areas of Operator Theory and Representation Theory. Consider a finite group $G$ of linear operators acting on $V$. For every nonzero $x \in V$ consider the related $G$-orbihedron $\operatorname{Co}_G x$ – the convex hull of the $G$-orbit of $x$. The convex geometry of $G$-orbihedra is important in numerous problems. In the case when $G$ is a Coxeter group, one can obtain very detailed information regarding the facial structure of $\operatorname{Co}_G x$ in convenient geometric terms – see [11] for the most comprehensive account. But as soon as we depart from Coxeter groups in their natural representations the situation becomes much more complicated. For example, the natural action of $G \times G$ on End $V$ by pre- and post-multiplications is not generated by reflections across hyperplanes, and all of the powerful machinery developed in [11] is not applicable. Moreover, preliminary computer experiments (C.K. Li, I. Spitkovsky, N. Zobin) show that the geometry of the related orbihedra may be very complicated. Nevertheless, there are several cases when it is possible to understand this geometry pretty well. It is more natural to consider a larger group $S_2^\otimes(G)$ generated by $G \times G$ and the operator $T \mapsto T^*$, where $T^*$ is the operator adjoint to $T$. First, $\operatorname{conv} G$ can be viewed as a $S_2^\otimes(G)$-orbihedron generated by the identity operator, and its facial structure does not seem too bad, at least in the case of a Coxeter group with a non-branching graph. The second example is the $S_2^\otimes(G)$-orbihedron generated by a Birkhoff tensor. One can easily see that the group $S_2^\otimes$ acts transitively on the set of Birkhoff tensors of a Coxeter group with a non-branching graph, so the set $\mathcal{B}_G$ is the set of extreme vectors of a $S_2^{\otimes}(G)$-orbihedron. Since $(\mathcal{B}_G)^\circ = \operatorname{conv} G$ in this case (not yet verified for $G = H_4$) then for $b \in \mathcal{B}_G$ we have $\operatorname{Extr}(\operatorname{Co}_{S_2^{\otimes}(G)} b)^\circ = G$. For what other elements $b \in \operatorname{End} V$ does the related $S_2^{\otimes}(G)$-orbihedron have a simple facial structure? This is a very interesting (but seemingly difficult) problem.

Let us remark that an analogous problem for infinite groups $O(V)$ and $U(V)$ of, respectively, orthogonal operators on a real Euclidean space $V$ and unitary operators on a complex Hermitian space $V$, is closely related to the theory of Schatten - von Neumann ideals, which has been studied in great depth (though in different terms). One can answer some of these questions by rethinking classical results in the geometric theory of Schatten - von Neumann ideals (see, e.g., [6]).

Another way to study the geometric structure of $\operatorname{conv} G$ is to explore the group of linear operators on $\operatorname{End} V$ preserving $\operatorname{conv} G$. This is a sort of a linear preserver problem, rather popular in Linear Algebra, see [13]. There was considerable progress in this direction recently, see [8]. A general type of answer is as follows: the only operators preserving $\operatorname{conv} G$ are the so-called rigid embeddings, i.e., operators of the type $\phi(A) = gAh$ or $\phi(A) = gA^*h$, where $g, h$ belong to the normalizer of $G$ in $O(V)$, and $gh \in G$. Such results were known for groups $O(V)$ ([14]) and $A_n$ ([9]). Rather unexpectedly, rigid embeddings are not the only operators preserving $\operatorname{conv} B_n$, see [8].

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## 2. A brief review of Coxeter groups

Let us now address several facts concerning the theory of Coxeter groups. For greater detail, consult [1], [3], or [7]. Let $G \subset \operatorname{End} V$ be a group. Then $G$ is a **Coxeter group** if it is finite, generated by orthogonal reflections across hyperplanes (containing the origin), and acts **effectively** (i.e., $gx = x$ for all $g \in G$ implies $x = 0$).

2.1. **Roots and weights.** Let $\mathcal{M}(G)$ denote the set of all **mirrors** — the hyperplanes in $V$ such that the orthogonal reflections across them belong to the group $G$. Mirrors split the space $V$ into connected components, whose closures are polyhedral cones. These cones are called **Weyl chambers**. It is known that Weyl chambers are actually **simplicial** cones, i.e., they have exactly $\dim V$ faces of codimension 1. These faces are called the **walls** of the chamber. The simpliciality of Weyl chambers implies that each Weyl chamber has exactly $\dim V$ **extreme rays**, and each extreme ray does not belong to exactly one wall of the chamber. Unit normals to the mirrors are called **roots**, the set of all roots is denoted by $\mathcal{R}_G$.

Fix a Weyl chamber $C$. Consider the roots $n_i(C)$, $1 \le i \le \dim V$, perpendicular to its walls, directed inwards with respect to the chamber. We call these **fundamental** or **simple roots**. It is known that the group $G$ is generated by reflections across the walls (i.e., across the mirrors containing the walls) of a Weyl chamber, i.e., by the operators $R_i = \mathbf{I} - 2n_i \otimes n_i$, $1 \le i \le \dim V$. Consider the vectors $\omega_j(C)$, $1 \le j \le \dim V$, such that $\langle n_i, \omega_j \rangle = c_j \delta(i-j)$, $c_j > 0$. These vectors are called the **fundamental weights** associated with the chamber $C$. The exact values of $c_j$ are not important for our purposes (for a standard normalization of fundamental weights see [3]). One can easily see that the fundamental weights are directed along the extreme rays of $C$. The set of all weights (i.e., those associated with any Weyl chamber) is denoted by $\mathcal{W}_G$. The sets $\mathcal{R}_G$ and $\mathcal{W}_G$ are fibered into $G$-orbits. It is known that the group $G$ acts **simply transitively** on the set of Weyl chambers (i.e., for every two chambers there exists exactly one element of $G$ transforming one of them onto the other). This immediately implies that the $G$-orbit of any vector $x$ (we denote it by $\operatorname{Orb}_G x$) intersects a Weyl chamber at exactly one point, let $x^*(C)$ denote the only point of $C \cap \operatorname{Orb}_G x$. Consider $m_G(x, y) = \max_{g \in G}\langle gx, y\rangle$. It is known (see [16]) that for any Weyl chamber $C$

$$m_G(x, y) = \langle x^*(C), y^*(C)\rangle \geq 0.$$

In fact, one can easily show that $m_G(x, y) > 0$ for irreducible Coxeter groups, provided $x, y$ are both nonzero.

2.2. **Coxeter graphs.** Since a Coxeter group $G$ is generated by reflections across the walls of any Weyl chamber, all information about the group is encrypted in the geometry of a Weyl chamber. In its turn the whole geometry of a Weyl chamber is described by the angles between its walls. Since the group is finite, these angles must be $\pi/k,\ k \in \mathbf{Z}_+, k \geq 2$. There is a wonderful way to encode the information about the angles in a graph. Let $\Gamma(G)$ denote the **Coxeter graph** of $G$, constructed as follows: the set $\operatorname{vert}(G)$ of vertices of the graph is in a one-to-one correspondence with the set of walls of a fixed Weyl chamber $C$, and two vertices are joined by an edge if and only if the angle between the related walls is $\pi/k,\ k \geq 3$, and $k-2$ is the multiplicity of the edge. Since the group acts transitively on the set of Weyl chambers, the Coxeter graph does not depend upon the choice of a Weyl chamber. Every fundamental root $n_i(C)$ is naturally associated with a wall of the Weyl chamber $C$, so it is associated with a vertex of $\Gamma(G)$. Every fundamental weight $\omega_j(C)$ is naturally associated with a unique wall of $C$ (namely, with the one it does not belong to), so it is naturally associated with a vertex $\pi$ of the graph $\Gamma(G)$. One can easily see that all weights from the same $G$-orbit are associated with the same vertex, so there is a one-to-one correspondence between the $G$-orbits of weights and the vertices of $\Gamma(G)$. Let $\pi(\omega)$ denote the vertex of $\Gamma(G)$ associated with the $G$-orbit of $\omega \in \mathcal{W}_G$.

An **end vertex** of the Coxeter graph $\Gamma(G)$ is any vertex connected to only one other vertex. A weight associated with an end vertex of $\Gamma(G)$ is called an **extremal weight**, the set of extremal weights is denoted by $E_G$.

A Coxeter graph is **branching** if it contains a vertex (called a **branching vertex**) connected to at least three other vertices. Otherwise, the graph is **non-branching**.

It is known that a Coxeter group $G$ is irreducible if and only if its Coxeter graph $\Gamma(G)$ is connected. All connected Coxeter graphs are classified, so all irreducible Coxeter groups are classified (see [3], [1], [7]). In particular, the classification shows that a connected Coxeter graph contains at most one branching vertex, the branching vertex is connected to exactly three other vertices, and all edges of a branching graph have multiplicity 1.

2.3. **Supports and stabilizers.** Fix a Weyl chamber $C$, let $\omega_i,\ 1 \leq i \leq \dim V$, denote the related fundamental weights, and for each $i,\ 1 \leq i \leq \dim V$, let $W_i$ denote the wall of $C$ not containing $\omega_i$. Since $C$ is a simplicial cone then for every

$a \in V$ there exists a unique decomposition

$$a^*(C) = \sum_i \lambda_i(a^*(C))\omega_i, \quad \lambda_i(a^*(C)) \geq 0.$$

Obviously,

$$\lambda_i(a^*(C)) = \frac{\langle a^*(C), n_i\rangle}{c_i}.$$

Introduce the **support** of $a$ as follows:

$$\operatorname{supp}_G a = \{\pi_i \in \operatorname{vert}(G) : \lambda_i(a^*(C)) > 0\} = \{\pi_i \in \operatorname{vert}(G) : a^*(C) \notin W_i\}.$$

One can easily show that $\operatorname{supp}_G a$ does not depend upon the choice of the chamber $C$, and it actually depends only upon the $G$-orbit of $a$.

For $a \in C$ let

$$\operatorname{Stab}_G a = \{g \in G : ga = a\}$$

It is well known (see, e.g., [3]) that this subgroup is generated by reflections across the walls $W_i$ of $C$, containing $a$. This subgroup is not a Coxeter group since the intersection of all mirrors containing $a$ (we denote this intersection by $V^a$) is a nontrivial subspace of fixed vectors. Let us restrict the action of this subgroup to its invariant subspace $V_a = (V^a)^\perp$. Thus, the nontrivial fixed vectors are cut off, and we get a Coxeter group

$$G_a = \operatorname{Stab}_G a|_{V_a}$$

acting on this subspace $V_a$. Its Coxeter graph can be computed as follows (see [16]):

$$\Gamma(G_a) = \Gamma(G) \setminus \operatorname{supp}_G a.$$

The latter means that all the vertices from $\operatorname{supp}_G a$ are erased, as well as all adjacent edges. So, if $\omega$ is a fundamental weight then $V_\omega = \omega^\perp$ and $\Gamma(G_\omega) = \Gamma(G) \setminus \{\pi(\omega)\}$. If $\omega$ is an extremal fundamental weight then the group $G_\omega$ acts irreducibly on $\omega^\perp$, since the graph $\Gamma(G) \setminus \{\pi(\omega)\}$ is connected.

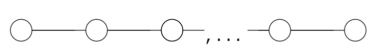
We recall the definitions of several Coxeter groups together with their extremal weights.

2.4. **Classification of irreducible Coxeter groups.** The following are descriptions of all finite irreducible Coxeter groups. First let us examine the four infinite families $A_n$, $B_n$, $D_n$, and $I_2(n)$.

Define $\mathrm{Perm}_{n+1}$ to be the group of linear operators acting on $\mathbf{R}^{n+1}$ by permutations of the canonical basis $\{e_1, e_2, e_3, ..., e_{n+1}\}$. We see that $e = \sum_{i=1}^{n+1} e_i$ is a fixed vector; now restrict the action to the invariant subspace $e^\perp$.

**Definition 2.1.** $A_n = \{T|_{e^\perp} : T \in \mathrm{Perm}_{n+1}\}$.

The related Coxeter graphs are:

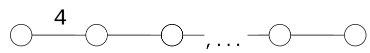


Vectors $\omega_1 = e_1 - e/(n+1)$, $\omega_n = e/(n+1) - e_{n+1}$ are extremal fundamental weights.

Note for future reference that $\omega_1$ is the orthogonal projection of the vector $e_1$ onto the subspace $e^\perp$. Also, $A_n$ does not contain $-\mathbf{I}$ (where $\mathbf{I}$ is the identity operator), and $\omega_n \in \operatorname{Orb}_{A_n}(-\omega_1)$.

**Definition 2.2.** *$B_n$ is the group of linear operators acting on $\mathbf{R}^n$ by taking $e_i$ to $p(i)e_{\sigma(i)}$, where $\sigma$ is a permutation of $\{1, ..., n\}$ and $p(i) = \pm 1$ for $1 \leq i \leq n$.*

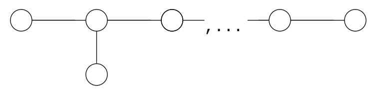
The related Coxeter graphs are:




The vectors $\omega_1 = e_1$, $\omega_n = e = e_1 + \cdots + e_n$ are extremal fundamental weights.

**Definition 2.3.** $D_n = \{T \in B_n : T$ *performs an* **even** *number of sign changes*$\}$.

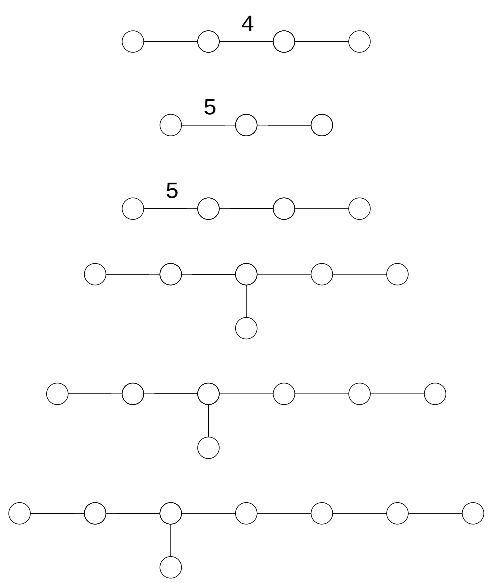
The related Coxeter graphs are:




**Definition 2.4.** *For* $n \geq 3$, $I_2(n)$ *is the dihedral group of order* $n$, *i.e., the group of symmetries of a regular* $n$*-gon. This is the group of operators acting on* $\mathbf{R}^2$ *generated by reflections across the lines* $y = 0$ *and* $y = tan(\pi/n)x$.

The related Coxeter graphs are:

Now, let us list the exceptional groups $F_4, H_3, H_4, E_6, E_7, E_8$. For these groups we give their Coxeter graphs only (in the order they are listed):




So, there exist four infinite families of irreducible Coxeter groups ($A_n, B_n, D_n, I_2(n)$) plus six exceptional groups ($E_6, E_7, E_8, F_4, H_3, H_4$). Each subscript indicates the dimension of the space $V$ where the group naturally acts. Coxeter graphs of $A_n, B_n, I_2(n), F_4, H_3, H_4$ are non-branching, Coxeter graphs of $D_n, E_6, E_7, E_8$ are branching.

## 3. The Main Theorem

**Theorem 3.1.** *(a) Let* $G = A_n, B_n, I_2(n), F_4$ *or* $H_3$*. Then*

$$\mathrm{Extr}\,(G^\circ) = \mathcal{B}_G.$$

*(b) Let* $G = D_n, E_6, E_7$ *or* $E_8$*. Then*

$$\mathrm{Extr}\,(G^\circ) \supsetneqq \mathcal{B}_G.$$

We prove the first assertion of this Theorem case by case. If $G = A_n$ or $B_n$, then the assertion is essentially the classical Birkhoff Theorem [2] – see details below. The case $G = I_2(n)$ was first proved in [10], but here we offer a much easier proof. The cases $G = F_4$ and $G = H_3$ are proved by computer calculations which we describe below. The second assertion — the case of the branching Coxeter graph — was proved in [10]; we reproduce the proof below.

## 4. Convex Geometry and Irreducible Coxeter Groups

As it has been already mentioned in the Introduction, we equip the space End $V$ with the scalar product $(T, S) =$ trace $(TS^*)$, and we identify $x \otimes y$ with the rank 1 operator $z \to x\langle z, y\rangle$. One can easily check that $(x \otimes y)^* = y \otimes x$, trace $(x \otimes y) = \langle x, y\rangle$, $(x \otimes y)(w \otimes t) = (x \otimes t)\langle y, w\rangle$.

4.1. **Polar Sets.** As usual, if we have a real Euclidean space $W$ with a scalar product $(.,.)$ then for a subset $U \subset W$ we consider its **polar set** $U^\circ = \{z \in W : \forall\, x \in U \quad (x, z) \le 1\}$. The set $U^\circ$ is a closed convex subset of $W$, containing 0. One can easily verify that $U^\circ = (\mathrm{conv}\, U)^\circ$. By the Bipolar Theorem, $(U^\circ)^\circ = \overline{\mathrm{conv}}\,(U \cup \{0\})$. So if $0 \in \mathrm{conv}\, G$ then $\mathrm{conv}\, G = ((\mathrm{conv}\, G)^\circ)^\circ$ (we may omit the closure since $\mathrm{conv}\, G$ is a closed polyhedron). We show that 0 is an **interior** point of $\mathrm{conv}\, G$, see Lemma 4.2. This implies that the set $(\mathrm{conv}\, G)^\circ$ is compact and therefore, by the Krein-Milman Theorem, this set is the closed convex hull of its extreme points. So the set $\mathrm{Extr}\,(\mathrm{conv}\, G)^\circ$ provides a nice description of the set $\mathrm{conv}\, G$ :

$$\mathrm{conv}\, G = (\mathrm{Extr}\,(\mathrm{conv}\, G)^\circ)^\circ = \{T \in \mathrm{End}\ V : (T, b) \le 1 \quad \forall\, b \in \mathrm{Extr}\,(\mathrm{conv}\, G)^\circ\}.$$

This formula and the definition of extreme points show that the elements of the set $\mathrm{Extr}\,(\mathrm{conv}(G)^\circ)$ are properly scaled normals to the faces of the polyhedron $\mathrm{conv}\, G$.

4.2. **Convex bodies associated with Coxeter groups.** All results of this subsection can be found in [10] – some without proofs.

Let us start with the following lemma which can be deduced from the Burnside Theorem, but we prefer to give a simple direct proof, especially because the idea is also used in the proof of Theorem 4.4.

**Lemma 4.1.** *Let* $G$ *be an irreducible Coxeter group. Then the set* $G$ *spans the whole space* End $V$.

*Proof.* Fix a Weyl chamber $C$. As before let $n_i$, $i = 1, 2, ..., \dim V$, denote the related fundamental roots (i.e., the **unit** normals to the walls of $C$), associated with the vertices $\pi_i$ of the Coxeter graph $\Gamma(G)$. These roots form a basis of $V$. The group $G$ is generated by the reflections $R_i = \mathbf{I} - 2n_i \otimes n_i$, $i = 1, 2, ..., \dim V$. Since $\mathbf{I} \in G$, all operators $n_i \otimes n_i$ are in span $G$. Considering the products $R_iR_j = \mathbf{I} - 2n_i \otimes n_i - 2n_j \otimes n_j + 4\langle n_i, n_j\rangle n_i \otimes n_j$ such that the vertices $\pi_i, \pi_j$ are connected by an edge (and therefore $\langle n_i, n_j\rangle \ne 0$), shows that all such operators $n_i \otimes n_j$ are in span $G$. Now choose three vertices $\pi_i, \pi_j, \pi_k$ such that the second one is connected by edges to the first and the third ones. Considering the product $R_iR_jR_k \in G$ and using the previous remarks, we show that $n_i \otimes n_k \in \operatorname{span} G$. Repeating the same trick, we show that all operators $n_i \otimes n_j$ are in span $G$, provided the vertices $\pi_i, \pi_j$ can be connected by a simple path in $\Gamma(G)$. Since the Coxeter graph of an irreducible group is connected, the Lemma is proven. ■

**Lemma 4.2.** *Let $G$ be an irreducible Coxeter group. Then* $0$ *is an interior point of the set* $\operatorname{conv} G$.

*Proof.* Consider the arithmetic mean $av_G$ of the elements of $G$. The group $G$ obviously fixes every element in the range of $av_G$, but this irreducible group cannot have nonzero fixed vectors, therefore $av_G = 0$. So, $0 = av_G \in \operatorname{conv} G$. Assuming that 0 is **not** an interior point of conv $G$, we find a nonzero operator $b \in \operatorname{End} V$ such that $(g, b) \leq 0$ for all $g \in G$. Therefore either $G \subset \{a \in \operatorname{End} V : (a, b) = 0\}$, or $(av_G, b) < 0$. The first is impossible because $G$ spans the space End $V$, and the second is impossible because $av_G = 0$. ■

This result implies that the set $G^\circ = (\operatorname{conv} G)^\circ$ is compact and therefore $G^\circ = \operatorname{conv} \operatorname{Extr} G^\circ$.

Noticing that the set $\mathcal{B}_G$ consists of rank 1 operators and is invariant under pre- and post-multiplications by operators from $G$ we arrive to the following result, which will be needed for Corollary 4.5:

**Corollary 4.3.** *Let $G$ be an irreducible Coxeter group. Then* $0 \in \operatorname{conv}(\mathcal{B}_G)$.

**Theorem 4.4.** $\mathcal{B}_G = (\operatorname{Extr} G^\circ) \cap (\text{ *rank 1 tensors* })$

*Proof.* One of the main results of [16] asserts that

$$\mathcal{B}_G = \operatorname{Extr} \operatorname{conv}(G^\circ \cap (\text{ rank 1 tensors })).$$

This is a rather deep result closely connected with the approach to interpolation of operators outlined in the Introduction.

Let us prove that $\mathcal{B}_G \subset \operatorname{Extr} G^\circ$. This will obviously imply the assertion of the Theorem.

Choose two extremal fundamental weights $\omega, \tau$ belonging to a Weyl chamber $C$, such that $\pi(\omega) \neq \pi(\tau)$. It suffices to show that $(\omega \otimes \tau)/m_G(\omega, \tau) \in \operatorname{Extr} G^\circ$. Consider the set $\mathcal{M} = \{g \in G : (g, \omega \otimes \tau) = \langle g\tau, \omega\rangle = m_G(\tau, \omega) = m_G(\omega, \tau)\}$. Since for any $g \in G$ we have $(g, \omega \otimes \tau) = \langle g\tau, \omega\rangle \leq m_G(\omega, \tau)$, $\operatorname{conv} \mathcal{M}$ is a face of $\operatorname{conv} G$ and all we need to show is that its dimension is maximal, i.e., to prove that $\mathcal{M}$ spans End $V$.

Define $\mathcal{P} = \{hg : h \in \operatorname{Stab}_G(\omega), g \in \operatorname{Stab}_G(\tau)\}$. Obviously, $\mathcal{P} \subset \mathcal{M}$.

Let $n_i,\ 1 \leq i \leq N = \dim V$, denote the fundamental roots associated with the chamber $C$; we assume that all roots are of unit length. Let $\omega_i,\ 1 \leq i \leq N$, denote the related fundamental weights, we assume that $\tau = \omega_1,\ \omega = \omega_N$. Let $R_j = \mathbf{I} - 2n_j \otimes n_j$ be the corresponding reflections. Recall that $\operatorname{Stab}_G(\omega_i)$ is generated by $\{R_j : j \neq i\}$.

Obviously, $\mathbf{I} \in \operatorname{Stab}_G(\omega_1) \bigcap \operatorname{Stab}_G(\omega_N)$. Also, note $R_j \in \operatorname{Stab}_G(\omega_1)$ for all $1 < j \leq N$, and $R_j \in \operatorname{Stab}_G(\omega_N)$ for all $1 \leq j < N$. Thus, for all $1 < j \leq N$, $n_j \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_1)$. Similarly, for all $1 \leq j < N$, $n_j \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_N)$.

Choose any $i, j$ such that $1 < i, j \leq N$. Let $\pi_i = \pi_{k_1}, \pi_{k_2}, \ldots, \pi_{k_r} = \pi_j$ be a simple path in $\Gamma(G)$, connecting $\pi_i$ to $\pi_j$. Such a path exists since $\Gamma(G)$ is connected. For all $1 \le l \le r$, we see that $k_l \neq 1$, so $n_{k_l} \otimes n_{k_l} \in \operatorname{span} \operatorname{Stab}_G(\omega_1)$. Then the product $(n_{k_1} \otimes n_{k_1})(n_{k_2} \otimes n_{k_2}) \dots (n_{k_r} \otimes n_{k_r})$ is also in $\operatorname{span} \operatorname{Stab}_G(\omega_1)$. Since

$$\begin{aligned}(n_{k_1} \otimes n_{k_1})&(n_{k_2} \otimes n_{k_2}) \dots (n_{k_r} \otimes n_{k_r}) \\ &= \langle n_{k_1}, n_{k_2}\rangle\langle n_{k_2}, n_{k_3}\rangle \dots \langle n_{k_{r-1}}, n_{k_r}\rangle (n_{k_1} \otimes n_{k_r})\end{aligned}$$

and for any $1 \le l < r$, $\langle n_{k_l}, n_{k_{l+1}}\rangle \neq 0$ (the vertices $\pi_{k_l}$ and $\pi_{k_{l+1}}$ are joined in $\Gamma(G)$), $n_{k_1} \otimes n_{k_r} = n_i \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_1)$ for all $1 < i, j \le N$. Repeating the same argument for $\operatorname{Stab}_G(\omega_N)$ yields $n_i \otimes n_j \in \operatorname{span} \operatorname{Stab}_G(\omega_N)$ for all $1 \le i, j < N$.

Choose $\pi_m$ adjacent to $\pi_1$. Now $(n_1 \otimes n_1)(n_m \otimes n_N) \in \operatorname{span}(\mathcal{P})$. Since $\langle n_1, n_m\rangle \neq 0$, $n_1 \otimes n_N \in \operatorname{span}(\mathcal{P})$.

To show that $n_N \otimes n_1 \in \operatorname{span}(\mathcal{P})$ requires a slightly more refined argument. Since the system $\{n_i : 1 \le i \le N\}$ is a basis in the space $V$, and the system $\{\omega_i : 1 \le i \le N\}$ is biorthogonal to this basis, then one can easily show that

$$\mathbf{I} = \sum_{i,j=1}^{N} \langle \omega_i, \omega_j\rangle n_j \otimes n_i$$

Notice that $\langle \omega_i, \omega_j\rangle \neq 0$ (in fact, $> 0$) for any $1 \le i, j \le N$, because $\omega_i$ and $\omega_j$ are in the same Weyl chamber, and $G$ is irreducible. For all $(i, j) \neq (N, 1)$, $n_i \otimes n_j \in \operatorname{span}(\mathcal{P})$, and $\mathbf{I} \in \operatorname{span}(\mathcal{P})$, so

$$\mathbf{I} - \sum_{\substack{1 \le i,j \le N \\ (i,j) \neq (N,1)}} \langle \omega_i, \omega_j\rangle n_i \otimes n_j \quad = \quad \langle \omega_n, \omega_1\rangle n_N \otimes n_1$$

is in $\operatorname{span}(\mathcal{P})$. Therefore, $n_N \otimes n_1 \in \operatorname{span}(\mathcal{P})$.

Thus, for all $1 \le i, j \le N$, $n_i \otimes n_j \in \operatorname{span} \mathcal{P}$. Since $\{n_i \otimes n_j : 1 \le i, j \le N\}$ form a basis for End $V$ and $\mathcal{P} \subset \mathcal{M}$, we see that $\mathcal{M}$ spans End $V$ as required. ■

The next result easily follows from the previous ones and the Bipolar Theorem:

**Corollary 4.5.** *The following are equivalent:*

(1) ${\mathcal{B}_G}^\circ \subset \operatorname{conv} G$.
(2) ${\mathcal{B}_G}^\circ = \operatorname{conv} G$.
(3) $\operatorname{Extr}(G^\circ) = \mathcal{B}_G$.
(4) $\operatorname{Extr}(G^\circ) \subset \mathcal{B}_G$.

## 5. The Proof of Theorem 3.1 for the groups $A_n$ and $B_n$

In this section we essentially reproduce the considerations from [10].

Birkhoff's theorem can be reformulated as follows:

**Theorem 5.1.** Extr $A_n^\circ = \mathcal{B}_{A_n}$.

*Proof.* Recall that $e = \sum_{i=1}^{n+1} e_i$. Definition 1.1 means that $T \in \Omega_{n+1}$ if and only if $Te = e$, $T^*e = e$ and $T$ transforms the positive orthant of $\mathbf{R}^{n+1}$ into itself. So, $e^\perp$ is invariant under $T \in \Omega_{n+1}$. Therefore $T$ transforms the intersection of the positive orthant with the affine hyperplane

$$\frac{1}{n+1} e + e^\perp$$

into itself. It is easy to see that this intersection is precisely $\operatorname{conv} \operatorname{Orb}_{\mathrm{Perm}_{n+1}} e_1$. Therefore $T$ also transforms the set $S$ – the orthogonal projection of this intersection onto the subspace $e^{\perp}$ – into itself. Since $\omega_1 = \operatorname{proj}_{d^{\perp}} e_1$ (see the description of $A_n$ in Section 2) then

$$S = \operatorname{proj}_{e^{\perp}} \operatorname{conv} \operatorname{Orb}_{\mathrm{Perm}_{n+1}} e_1 = \operatorname{conv} \operatorname{Orb}_{A_n} \operatorname{proj}_{e^{\perp}} e_1 = \operatorname{conv} \operatorname{Orb}_{A_n} \omega_1$$

It is known (see [16]) that

$$\operatorname{Extr} (\operatorname{Orb}_{A_n} \omega_1)^{\circ} = \frac{1}{m_G(\omega_1, \omega_n)} \operatorname{Orb}_{A_n} \omega_n.$$

So we conclude that $TS \subset S$ if and only if $(T, h\omega_n \otimes g\omega_1) = \langle Tg\omega_1, h\omega_n \rangle \leq m_G(\omega_1, \omega_n)$ for all $g, h \in A_n$. Since $\omega_1 \in \operatorname{Orb}_{A_n}(-\omega_n)$ and $\omega_n \in \operatorname{Orb}_{A_n}(-\omega_1)$, the sets $\{h\omega_n \otimes g\omega_1 : g, h \in A_n\}$ and $\{g\omega_1 \otimes h\omega_n : g, h \in A_n\}$ coincide. Therefore $T \in \Omega_{n+1}$ if and only if $Te = e, Te^{\perp} \subset e^{\perp}$ and $T|_{e^{\perp}} \in (\mathcal{B}_{A_n})^{\circ}$. This means that $\operatorname{Extr} (\Omega_{n+1}|_{e^{\perp}})^{\circ} \subset \mathcal{B}_{A_n}$. By the Birkhoff Theorem, $\operatorname{Extr} \Omega_{n+1} = \mathrm{Perm}_{n+1}$, so $(\Omega_{n+1})^{\circ} = (\mathrm{Perm}_{n+1})^{\circ}$ and, since both $\Omega_{n+1}$ and $\mathrm{Perm}_{n+1}$ leave $e^{\perp}$ invariant, we get

$$\operatorname{Extr} A_n^{\circ} = \operatorname{Extr} (\mathrm{Perm}_{n+1}|_{e^{\perp}})^{\circ} = \operatorname{Extr} (\Omega_{n+1}|_{e^{\perp}})^{\circ} \subset \mathcal{B}_{A_n}.$$

According to Lemma 4.5, this proves the result. ■

**Definition 5.2.** *An $n \times n$ matrix $(a_{ij})$ is called* **absolutely bistochastic** *if*

$$\text{for every } j, \ 1 \leq j \leq n, \qquad \sum_{i=1}^{n} |a_{ij}| \leq 1, \qquad \sum_{i=1}^{n} |a_{ji}| \leq 1.$$

*Let $\mho_n$ be the set of all absolutely bistochastic $n \times n$ matrices.*

The next Lemma follows from the Birkhoff Theorem — see, for example, [12].

**Lemma 5.3.** $B_n = \operatorname{Extr} (\mho_n)$

The desired description is now (almost) immediate.

**Theorem 5.4.** $\operatorname{Extr} B_n^{\circ} = \mathcal{B}_{B_n}$.

*Proof.* By Lemma 4.5, it suffices to prove $(\mathcal{B}_{B_n})^{\circ} \subset \operatorname{conv}(B_n)$. However, by Lemma 5.3, this statement is equivalent to $(\mathcal{B}_{B_n})^{\circ} \subset \mho_n$. Let $A = (a_{ij}) \in (\mathcal{B}_{B_n})^{\circ}$. Let $q = \sum_{j=1}^{n} \varepsilon_j e_j, \quad \varepsilon_j = \pm 1$. All such $q$ form the $B_n$-orbit of the extremal fundamental weight $\omega_n$. Then $(A, q \otimes e_i) = \langle Ae_i, q \rangle \leq 1$ for all $q \in Q, 1 \leq i \leq n$. This is equivalent to $\sum_{j=1}^{n} \varepsilon_j a_{ij} \leq 1$ for all $\varepsilon_j = \pm 1$, $1 \leq i \leq n$, or $\sum_{j=1}^{n} |a_{ij}| \leq 1$. Similarly, using $(A, e_i \otimes q)$, deduce $\sum_{i=1}^{n} |a_{ij}| \leq 1$. So $A \in \mho_n$. ■

## 6. A proof of Theorem 3.1 for $I_2(n)$.

Recall that the group $I_2(n)$ is a dihedral group acting on $\mathbf{R}^2$, i.e., the group of symmetries of a regular $n$-gon, with one vertex on the positive $x$-axis.

Let $\operatorname{Rot}(\theta)$ be the linear operator performing counter-clockwise rotation by the angle $\theta$. Let

$$\operatorname{Refl}(\theta) = \operatorname{Rot}(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \operatorname{Rot}(-\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

be the linear operator performing reflection across the line at an angle $\theta$ from the $x$-axis in the counter-clockwise direction.

One can easily see that every orthogonal operator in $\mathbf{R}^2$ is either a rotation, or a reflection, depending upon whether its determinant is $+1$ or $-1$. (Indeed, its eigenvalues are either a pair of mutually conjugate complex numbers on the unit circle, or they are $\pm 1$; in the first case the determinant is 1 and this operator is obviously a rotation, in the second case the determinant is $-1$ and the operator is obviously a reflection.)

Let

$$\mathrm{Rot}_n = \{\mathrm{Rot}\,(2\pi k/n) : 0 \le k < n\}, \quad \mathrm{Refl}_n = \{\mathrm{Refl}\,(\pi k/n) : 0 \le k < n\}.$$

Obviously, $I_2(n) \supset \mathrm{Rot}_n \cup \mathrm{Refl}_n$. Actually, these two sets coincide — a regular $n$-gon is not invariant under other rotations and reflections.

**Lemma 6.1.**

$$I_2(n) = \mathrm{Rot}_n \cup \mathrm{Refl}_n\,.$$

Each of the sets $\mathrm{Rot}_n$, $\mathrm{Refl}_n$ spans a two-dimensional subspace in $\mathrm{End}\,(\mathbf{R}^2)$. There are exactly $n$ elements in each of these two sets, and they are equidistributed on the unit circles in the related subspaces. So, $\mathrm{conv}\,\mathrm{Refl}_n$ and $\mathrm{conv}\,\mathrm{Rot}_n$ are regular $n$-gons in the related two-dimensional subspaces of the four-dimensional space $\mathrm{End}\,\mathbf{R}^2$.

The following Lemma is easily verified.

**Lemma 6.2.** *The two-dimensional subspaces spanned by* $\mathrm{Rot}_n$ *and* $\mathrm{Refl}_n$ *are mutually orthogonal.*

**Corollary 6.3.** *Every face* $\Phi$ *of* $\mathrm{conv}\, I_2(n)$ *of maximal dimension is uniquely representable as*

$$\Phi = \mathrm{conv}(\phi \cup \psi)$$

*where* $\phi$ *is a side of the regular* $n$*-gon* $\mathrm{conv}\, Rot_n$*, and* $\psi$ *is a side of the regular* $n$*-gon* $\mathrm{conv}\, Refl_n$*.*

*Conversely, for any two sides* $\phi \subset \mathrm{conv}\, Rot_n$ , $\psi \subset \mathrm{conv}\, Refl_n$ *the set* $\mathrm{conv}(\phi \cup \psi)$ *is a face of* $\mathrm{conv}\, I_2(n)$ *of maximal dimension.*

*Proof.* Let $\Phi$ be a face of $\mathrm{conv}\, I_2(n)$ of maximal dimension. Then all operators from $I_2(n)$ are in a half-space defined by the hyperplane containing $\Phi$. This hyperplane does not contain the origin, therefore it does not contain $\mathrm{span}\,\mathrm{Rot}_n$ or $\mathrm{span}\,\mathrm{Refl}_n$. Therefore its intersections with these subspaces are hyperplanes in these subspaces. $\Phi$ must contain four linearly independent elements of $I_2(n)$. Since a hyperplane in a two-dimensional subspace can contain no more that two linearly independent elements, $\Phi$ must contain exactly two linearly independent rotations and exactly two linearly independent reflections. Since $\mathrm{conv}\,\mathrm{Rot}_n$ and $\mathrm{conv}\,\mathrm{Refl}_n$ are in a half-space defined by the hyperplane, $\Phi$ contains sides $\phi$ and $\psi$ of the regular $n$-gons $\mathrm{conv}\,\mathrm{Rot}_n$ and $\mathrm{conv}\,\mathrm{Refl}_n$, respectively. Thus $\Phi = \mathrm{conv}(\phi \cup \psi)$.

Now let $\phi$ and $\psi$ be sides of the convex $n$-gons $\mathrm{conv}\,\mathrm{Rot}_n$ and $\mathrm{conv}\,\mathrm{Refl}_n$, respectively. Since these convex sets are in mutually orthogonal subspaces, $\mathrm{conv}(\phi \cup \psi)$ contains four linearly independent elements of $I_2(n)$, so it defines a face of $\mathrm{conv}\, I_2(n)$ of maximal dimension. ■

**Corollary 6.4.** $\mathrm{conv}\, I_2(n)$ *has exactly* $n^2$ *faces of maximal dimension.*

**Lemma 6.5.** $\mathrm{card}\,(\mathcal{B}_{I_2(n)}) = n^2$.

*Proof.* Obviously, the cone bounded by the lines $y = 0$ and $y = \tan(\pi/n)x$ is a Weyl chamber for $I_2(n)$. Therefore the vectors $\omega_1 = (1, 0)$ and $\omega_2 = (\cos(\pi/n), \sin(\pi/n))$ are the (extremal) fundamental weights. Therefore

$$\mathcal{B}_{I_2(n)} = \{g\omega_1 \otimes h\omega_2, h\omega_2 \otimes g\omega_1 : g, h \in I_2(n)\}.$$

Note that $\mathrm{card}\,(\mathrm{Orb}_{\,I_2(n)}\,\omega_i) = n$ for $i = 1, 2$. If $n$ is even then $-\mathbf{I} \in I_2(n)$ and since $(-\omega) \otimes (-\tau) = \omega \otimes \tau$ we get $\mathrm{card}\,(\mathcal{B}_{I_2(n)}) = (2)(n)(n)/2 = n^2$. If $n$ is odd then $-\mathbf{I} \notin I_2(n)$, and $-\omega_1 \in \mathrm{Orb}_{\,I_2(n)}\,\omega_2$. So again $\mathrm{card}\,(\mathcal{B}_{I_2(n)}) = (2)(n)(n)/2 = n^2$. ■

So, the number of Birkhoff tensors $\mathrm{card}\,(\mathcal{B}_{I_2(n)})$ equals the overall number of faces of $\mathrm{conv}\, I_2(n)$. Using Theorem 4.4, we arrive at the following result, which proves Theorem 3.1 for $G = I_2(n)$.

**Corollary 6.6.** $\mathrm{Extr}\,(I_2(n))^\circ = \mathcal{B}_{I_2(n)}$.

This result was obtained in [10] by lengthy computations.

## 7. Extreme elements of $(D_4)^\circ$

**Theorem 7.1.**

$$\mathrm{Extr}\,(D_4)^\circ = \mathcal{B}_{D_4} \bigcup \{gAh : g, h \in D_4\},$$

*where*

$$A = \frac{1}{4}\begin{pmatrix} -2 & 2 & 0 & -1 \\ 2 & -2 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This result was obtained in [10] by a computer calculation (in exact arithmetic), using the cdd program, written by Komei Fukuda [4]. We discuss this program below in Section 10.

Obviously, the matrix $A$ is of rank 3, so $\mathcal{B}_{D_4} \subsetneqq \mathrm{Extr}\,(D_4)^\circ$. Actually what we need is the existence of a matrix of rank greater than 1 in $\mathrm{Extr}\,(D_4)^\circ$. It is not hard to check by hand that the matrix $A$ belongs to $\mathrm{Extr}\,(D_4)^\circ$, i.e., to verify that the scalar product of $A$ with every element of $D_4$ does not exceed 1 and to explicitly find 16 linearly independent elements of $D_4$ whose scalar products with this matrix are exactly 1. So, the proof of the Conjecture for the group $D_4$ does not formally depend upon the use of computer calculations.

## 8. Coxeter groups with branching graphs

In this Section we prove Part (b) of the Conjecture, see [10].

**Theorem 8.1.** *Let $G$ be a finite irreducible Coxeter group with a branching Coxeter graph $\Gamma(G)$. Then not all elements of* $\mathrm{Extr}\,(\mathrm{conv}\, G)^\circ$ *are of rank* 1*, i.e.,* $\mathcal{B}_G \subsetneqq \mathrm{Extr}\,(\mathrm{conv}\, G)^\circ$.

*Proof.* It is known from the classification of connected Coxeter graphs (see, e.g., [3]) that every branching Coxeter graph contains a (branching connected) graph $\Gamma(D_4)$ as a subgraph. The statement of Theorem is valid for this group — see the previous Section. So we may assume that $\Gamma(G) \neq \Gamma(D_4)$. Therefore there exists an end vertex $\pi$ such that the graph $\Gamma(G) \setminus \{\pi\}$ is a branching connected Coxeter graph.

**Claim.** If all elements of $\operatorname{Extr}(\operatorname{conv} G)^\circ$ are of rank 1, then the same is true for $\operatorname{Extr}(\operatorname{conv} H)^\circ$ where $H$ is a Coxeter group such that $\Gamma(H) = \Gamma(G) \setminus \{\pi\}$.

This claim, together with the above considerations, easily leads to a proof of the Theorem.

Let all elements of $\operatorname{Extr}(\operatorname{conv} G)^\circ$ be of rank 1. Let $\omega$ be an extremal fundamental weight associated with the vertex $\pi$. We may assume that its length is 1.

Consider the Coxeter group $G_\omega = \operatorname{Stab}_G \omega|_{\omega^\perp}$ and denote it by $H$. Then

$$\Gamma(H) = \Gamma(G) \setminus \{\pi\}.$$

Since $\pi$ is an end vertex, this graph is connected and therefore $H$ is an irreducible group. Consider the hyperplane $\Pi = \{T \in \operatorname{End} V : (T, \omega \otimes \omega) = 0\}$. Note that

$$\operatorname{Stab}_G \omega = G \cap (\Pi + \mathbf{I})$$

This immediately follows from the fact that the elements of $G$ are all orthogonal operators. Also, the affine hyperplane $(\Pi + \mathbf{I})$ is a support hyperplane of the polyhedron $\operatorname{conv} G$, i.e.,

$$G \subset \{T \in \operatorname{End} V : (T, \omega \otimes \omega) \leq \langle \omega, \omega \rangle\}$$

Therefore the faces of maximal dimension of the polyhedron $\operatorname{conv}(\operatorname{Stab}_G \omega)$ are intersections of faces of $\operatorname{conv} G$ with the hyperplane $\Pi + \mathbf{I}$. We have assumed that the normals to all faces of $\operatorname{conv} G$ are of rank 1. We obtain the group $H$ from the group $\operatorname{Stab}_G \omega$ by restricting the action of the latter to its invariant subspace $\omega^\perp$. We can view this as follows:

Let $P$ denote the orthogonal projection of $V$ onto the subspace $\omega^\perp$. Then the operator $T \mapsto PTP$ is an orthogonal projection in End $V$. Then $H = P(\operatorname{Stab}_G \omega)P$. Therefore the faces of maximal dimension of $\operatorname{conv} H$ are projections of the faces of maximal dimension of $\operatorname{conv}(\operatorname{Stab}_G \omega)$. Thus the normals to faces of $\operatorname{conv}(H)$ are of the form $PbP$, where $b \in \operatorname{Extr}(\operatorname{conv} G)^\circ$. But all these tensors are of rank 1. So the Claim is proven, which completes the proof of Theorem. ■

## 9. Computer tools

The results of the previous sections leave the following exceptional groups for which Conjecture 1.4 still needs verification: namely the groups $F_4, H_3$ and $H_4$. In this Section we discuss computer tools which have enabled us to verify the Conjecture for $G = F_4, H_3$, and we also discuss some approaches which hopefully will in the future allow us to verify the Conjecture for the remaining case $G = H_4$. All programs we have written are available at http://www.math.wm.edu/ zobin/. Our main tool is a cdd program which calculates the extreme elements of a polytope given by a system of linear inequalities, which is exactly what $G^\circ$ is. We must be able to write down the system of inequalities as an input file, so we need to obtain a list of matrices corresponding to the operators which are the elements of our group $G$. This is the first computational problem we address.

9.1. **Matrix Representation of Coxeter Groups.** All information about a Coxeter group is encoded in its graph, but going from the graph to a presentation of elements is not easy, and the computer can help here.

We wrote a program in C++ which, given a Coxeter graph, lists the matrices of all elements of the associated group in a natural orthonormal basis, together with Birkhoff tensors, fundamental roots and weights, generators, etc. To explain this program, we'll follow its logic and note its output for $H_3$. We assume that the input graph has $n$ vertices, labeled with the associated fundamental weights $\omega_1, \omega_2, \ldots, \omega_n$.

Although the program takes a graph as its input, the computer is happier working with a matrix. Thus, the computer represents the given graph as a Cartan matrix:

**Definition 9.1.** *Consider the Coxeter graph $\Gamma(G)$ on $n$ vertices. Choose an ordering $\preccurlyeq$ of the vertices (for a non-branching graph there are two natural orderings of the vertices, going along the path). So, label the vertices $\pi_1, \pi_2, \cdots, \pi_n$ and assume that as $i$ goes from 1 to $n$ the vertices are arranged according to the chosen ordering. Then we have an ordering $W_1, W_2, \cdots, W_n$ of the walls in a Weyl chamber, as well as an ordering $r_1, r_2, \cdots, r_n$ of the fundamental roots. The* **modified Cartan matrix** *$\mathfrak{C}(G, \preccurlyeq)$ associated with this ordering of the fundamental roots is the $n \times n$ matrix $(a_{ij})$ with*

$$a_{ij} = -\cos(\pi/(k(i,j)+2)) = \langle r_i, r_j \rangle$$

*for $i \neq j$, and $a_{ii} = 1 = \langle r_i, r_i \rangle$. Here $k(i,j)$ is the multiplicity of the edge joining the vertices $\pi_i$ and $\pi_j$ of the graph, and $r_i$ are the fundamental roots. So,*

$$\mathfrak{C}(G, \preccurlyeq) = (\langle r_i, r_j \rangle)_{1 \leq i,j \leq n}.$$

For instance, choosing the ordering from the left to the right for the vertices of $\Gamma(H_3)$ (see 2.4) we get

$$\mathfrak{C}(H_3, \preccurlyeq) = \begin{pmatrix} 1 & -\cos(\pi/5) & 0 \\ -\cos(\pi/5) & 1 & -\cos(\pi/3) \\ 0 & = \cos(\pi/3) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -(1+\sqrt{5})/4 & 0 \\ -(1+\sqrt{5})/4 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}.$$

The natural orthonormal basis $e_1, e_2, \ldots, e_n$ which we are going to use is obtained from the basis of fundamental roots $r_1, r_2, \cdots, r_n$ by the Gram–Schmidt orthogonalization procedure.

Clearly then

$$\begin{aligned} r_1 &= \lambda_{11} e_1, \\ r_2 &= \lambda_{21} e_1 + \lambda_{22} e_2, \\ &\vdots \\ r_n &= \lambda_{n,1} e_1 + \cdots + \lambda_{n,n} e_n, \end{aligned}$$

for some $\lambda_{ij}$. It is, in fact, these scalars $\lambda_{ij}$ that we are after. These scalars form a lower-triangular matrix $\Lambda$. Obviously, $\lambda_{11} = 1$.

One can immediately see that

$$\Lambda \Lambda^t = (\langle r_i, r_j \rangle)_{1 \leq i,j \leq n} = \mathfrak{C}(G, \preccurlyeq).$$

So, the matrix $\Lambda$ is nothing else but the Cholesky factor of $\mathfrak{C}(G, \preccurlyeq)$. There are numerous programs for efficient Cholesky factorization.

For $H_3$, this gives us approximately

$$\Lambda(H_3) \approx \begin{pmatrix} 1 & 0 & 0 \\ 0.809\ldots & 0.588\ldots & 0 \\ 0 & 0.851\ldots & 0.526\ldots \end{pmatrix}.$$

Next, the program computes the fundamental weights. The fundamental weights form a basis dual to the basis of the fundamental roots. So if $W(G, \preccurlyeq)$ is the "matrix of fundamental weights" of $G$ in which each column gives the coordinates of a fundamental weight in the basis $e_1, e_2, \cdots, e_n$, then $W(G, \preccurlyeq) = (\Lambda^{-1})^t$. This is an easy calculation for the program to perform. The program gives

$$W(H_3, \preccurlyeq) \approx \begin{pmatrix} 1 & -1.376\ldots & 2.227\ldots \\ 0 & 1.701\ldots & -2.753\ldots \\ 0 & 0 & 1.902\ldots \end{pmatrix}.$$

Now we want the matrices for the elements of $G$. Getting a representation requires generators; consider the generators $R_i = \mathbf{I} - 2r_i \otimes r_i$. Their matrices in our basis are $g_i = \mathbf{I} - 2\Lambda_i^t \Lambda_i$ where $\Lambda_i$ the $i$-th row of the matrix $\Lambda$.

Generating all the elements requires iterating. We initialize a list with the $n$ generators. At each iteration, multiply each element in the list with every other element in the list; if the product is not in the list, add the new matrix to the list. If at any iteration, no new matrices are created, then stop.

This naïve algorithm generates the elements of $H_3$ quite quickly, but it takes too long to enumerate the elements of the much larger group $H_4$. A more intelligent approach is required; we can keep the previous method, but we have to reduce the number of spurious matrix multiplications. With each element in the list, store the **length** of the element, i.e., the minimal number of generators whose product is the element. Clearly, for the generators themselves, this number is 1. On the $k$-th iteration, then, we simply take all elements of length $k$, and pre- and post-multiply them by the generators to get all elements of length $k+1$. This is a pretty simple optimization, but it pays off quite well.

We intend to use this program to compile the input file for the cdd program, which actually computes the extreme elements of $G^\circ$.

With a list of elements in hand, writing down the Birkhoff tensors is easy. Since we've generated the fundamental weights, we can find a particular Birkhoff tensor $\omega \otimes \tau$. To find the others, we iterate through all $g, h \in G$ and add $g\omega \otimes h\tau$ to the list of tensors. Each time we add a tensor to the list, however, the program must check that the new tensor is in fact different from all the previously generated tensors.

Now that we can explore $H_3$ with the computer, we are ready to tackle the conjecture.

## 10. Computer Proofs of the Conjecture for $F_4$ and $H_3$.

We want to show that $\operatorname{Extr} G^\circ = \mathcal{B}_G$ for $G = F_4, H_3$. We'll verify this by finding $\operatorname{Extr} G^\circ$ with the aid of a computer. The algorithm we employ is the Double Description Method, otherwise known as Chernikova's algorithm. A (non-computer) calculation based on this algorithm was used in [18] for a solution of a problem regarding the geometry of some orbihedra, see also [11] for another approach to this problem.

Our exposition of this algorithm follows the one given in [5].

10.1. **The Double Description Method.** Given a finite set $S$ in $\mathbf{R}^n$, we want to find $\operatorname{Extr} S^\circ$. First we homogenize the problem by switching from $S$, to the cone $S' = \{\lambda(\{1\} \times S) \subset \mathbf{R}^{n+1} : \lambda \geq 0\}$, and from the polar set $S^\circ$ to the **dual cone**

$$D(S') = \{v \in \mathbf{R}^{n+1} : \forall s \in S',\ \langle s, v\rangle \geq 0\}.$$

Then the projections of extreme rays of the cone $P(S')$ to $\mathbf{R}^n$ are directed along the vectors from Extr $S^\circ$, so the problem of finding the convex hull is merely a disguise for the problem of enumerating the extreme rays of a polyhedral cone with vertex at the origin.

The algorithm we use takes a polyhedral cone given by a system of homogeneous linear inequalities and finds its extreme rays. As input, we have a matrix $A$, whose rows are the coefficients of the linear inequalities defining the cone, in other words, the normals to the faces of the cone. The matrix $A$ describes a cone

$$P(A) = \{v \in \mathbf{R}^{n+1} : Av \geq 0\}.$$

As output, we want a $(n+1) \times m$ matrix $R$ whose columns are vectors whose linear combinations with non-negative coefficients give the whole cone $P(A)$. So, $R$ provides an alternative description of this cone

$$P(A) = \{v \in \mathbf{R}^{n+1} : \exists \lambda \in \mathbf{R}^m, \lambda > 0,\, v = R\lambda\}.$$

The ordered pair $(A, R)$ is called a **double description** pair; such a pair provides two descriptions of the same cone.

Here of course there is a slight abuse of terminology, since we are identifying matrices with the (non ordered) sets of their column or row vectors.

The columns of $R$ include vectors directed along all of the extreme rays. But $R$ may also have other redundant columns. Clearly there exist many different matrices $R$ which form a double description pair with $A$.

The redundancy may be eliminated rather simply — we must omit all vectors belonging to less than $n$ independent boundary hyperplanes, i.e., turning less than $n$ independent inequalities (given by the rows of $A$) into equalities. The obtained matrix is already unique — up to permutations of columns and scaling of each column.

Simply put, the algorithm takes $A$ and finds a double description pair $(A, R)$. The method is iterative. Let $A_k$ be the matrix of the first $k$ rows of $A$. Since $A_{n+1}$ is a simplicial cone, finding a matrix $R_{n+1}$ is easy: just solve $A_{n+1}R_{n+1} = I$ to get a double description pair $(A_{n+1}, R_{n+1})$. Now iterate: suppose that for some $A_k$ we have a double description pair $(A_k, R_k)$; we'd like to find an $R_{k+1}$ for $A_{k+1}$. Let $a_{k+1}$ be the $(k+1)$-st row vector of $A$; this vector determines a hyperplane cutting $\mathbf{R}^{n+1}$ into three pieces (two half-spaces and a hyperplane) as follows:

$$\begin{aligned} H^+_{k+1} &= \{v \in \mathbf{R}^{n+1} : \langle v, a_{k+1}\rangle > 0\}, \\ H^0_{k+1} &= \{v \in \mathbf{R}^{n+1} : \langle v, a_{k+1}\rangle = 0\}, \text{ and} \\ H^-_{k+1} &= \{v \in \mathbf{R}^{n+1} : \langle v, a_{k+1}\rangle < 0\}. \end{aligned}$$

Let $r_1, \ldots, r_j$ be the columns of $R_k$. The vector $a_{k+1}$ will also partition these rays into three sets

$$\begin{aligned} J^+_{k+1} &= \{r_i : r_i \in H^+_{k+1}\}, \\ J^0_{k+1} &= \{r_i : r_i \in H^0_{k+1}\}, \text{ and} \\ J^-_{k+1} &= \{r_i : r_i \in H^-_{k+1}\}. \end{aligned}$$

What is the relationship between $R_{k+1}$ and $R_k$? Clearly, $J^+_{k+1} \subset R_{k+1}$ and $J^0_{k+1} \subset R_{k+1}$, but there might be something we are missing. Indeed it is not hard to prove the following lemma (see [5]):

**Lemma 10.1.** *Let $(A_k, R_k)$ be a double description pair. Then so is $(A_{k+1}, R_{k+1})$ where*
$$R_{k+1} = J^+_{k+1} \cup J^0_{k+1} \cup M,$$
*and*
$$M = \{\operatorname{span}\{u, v\} \cap H^0_{k+1} : u \in J^-_{k+1}, v \in J^+_{k+1}\}.$$

By applying this lemma iteratively, we eventually find a matrix $R$ forming a double description pair $(A, R)$. As it was mentioned earlier, this $R$ might include some extraneous rays; to ensure minimality of $R_{k+1}$ at each step with this naïve approach, we assume that $R_k$ is not redundant, and then we remove rays in $R_{k+1}$ that lie on fewer than $n$ hyperplanes of $A_{k+1}$.

It is always better if such a program works incrementally, outputting the extreme rays it has already calculated. This can be done by verifying feasibility of the vectors from the non-redundant version of $R_k$, i.e., checking if they satisfy **all** inequalities defining the cone. Then the feasible vectors should be listed and outputted as a partial result. Note that non-feasible vectors from $R_k$ should not be included in this partial list, but they are needed to perform the next iteration.

To actually perform this iterative process by computer, we use the cdd program, an implementation in C++ of the double description algorithm [4]. Although cdd is essentially the algorithm presented above, it is packed with many optimizations. These optimizations are not enough to make cdd the best ray enumeration algorithm for all problems—there are more efficient algorithms for simplicial polyhedra—but cdd is excellent for the degenerate (i.e., non-simplicial) case. Since the convex hulls of Coxeter groups are degenerate polyhedra in operator space, cdd is particularly well suited for finding Extr $G^\circ$ for $G$ a Coxeter group.

10.2. **Computing the convex hulls of $F_4$ and $H_3$.**. To compute Extr $G^\circ$ with the help of cdd we first need to prepare an input file, where we write down the system of linear inequalities describing $G^\circ$ :
$$\sum_{1\le i,j\le n} s_{ij} g_{ji} \le 1, \;\; g \in G.$$
Preparing such an input file is not too hard for relatively small groups like $F_4$ and $H_3$, but it is not at all easy for $H_4$, consisting of $14,400$ elements. So, we have compiled the input files for $F_4$ and $H_3$ by hand, and for the case of $H_4$ we have used the input file compiled by Val Spitkovsky, who has applied quite sophisticated programming tools to do this.

In the future we plan to generate these files with the help of our program listing the matrices $(g_{ij})$ of all operators from the group $G$. This could be important for verification of the Conjecture for the group $H_4$, since we hope to introduce some additional optimization into the cdd program (exploiting symmetries, a clever choice of the ordering, etc) and we shall need a significant flexibility in preparing the input file.

The double description method can be performed exactly in arithmetic over $\mathbf{Q}$. Since the matrices representing operators from the Coxeter group $F_4$ in our basis $e_1, e_2, e_3, e_4$ have rational entries, cdd can find Extr $(F_4)^\circ$ exactly.

**Theorem 10.2.** Extr $(F_4)^\circ = \mathcal{B}_{F_4}$.

The situation is more complicated for $H_3$: there is no basis in which the matrices for the elements of $H_3$ have rational entries (this is equivalent to the fact that $H_3$

and $H_4$ are not **crystallographic** groups, see [3]). Nonetheless, the matrices of the elements of $H_3$ in our basis $e_1, e_2, e_3$ are over the field $\mathbf{Q}(\sqrt{5})$, the algebraic extension of $\mathbf{Q}$ by $\sqrt{5}$.

To capitalize on this fact, we extended cdd to perform exact arithmetic over $\mathbf{Q}(\sqrt{5})$. Such arithmetic is easy to work with: an element of $\mathbf{Q}(\sqrt{5})$ is identified with an ordered pair $(p, q)$ where $p, q \in \mathbf{Q}$, i.e., $(p, q) \cong p + q\sqrt{5}$. Elementary algebra quickly verifies the following:

$$\begin{aligned} (p,q) + (p',q') &= (p+p', q+q'), \\ (p,q)(p',q') &= (pp' + 5qq', pq' + p'q), \\ (p',q')^{-1} &= \left( \frac{p}{p^2 - 5q^2}, \frac{-q}{p^2 - 5q^2} \right). \end{aligned}$$

Our modified version of cdd computes $\operatorname{Extr}(H_3)^\circ$ exactly over $\mathbf{Q}(\sqrt{5})$. We did this in under a half hour on a Pentium III. By Theorem 4.4,

$$\mathcal{B}_G = (\text{Extr } G\circ) \cap (\text{rank 1 tensors}).$$

So it suffices to verify that for all $v \in \operatorname{Extr}(H_3)^\circ$, $\operatorname{rank}(v) = 1$. The cdd output verified this, thereby proving

**Theorem 10.3.** $\operatorname{Extr}(H_3)^\circ = \mathcal{B}_{H_3}$.

## 11. Computer Attacks on $H_4$.

The matrices of operators from $H_4$ also belong to the field $\mathbf{Q}(\sqrt{5})$. Although in theory cdd can compute $\operatorname{Extr}(H_4)^\circ$, we can't even come close in practice. The group $H_3$ has only 120 elements; $H_4$ has $14,400$. The initial problem is memory: as cdd iterates, it generates a plethora of extraneous rays. After a couple of hundred iterations, the number of extraneous rays easily fills up all of memory.

However, our problem has a lot of symmetries and it is natural to try to use these symmetries to reduce the volume of computations. Here we discuss some steps which we have already taken in this direction and others which we hope to carry out in the future

Since the group acts by multiplications on itself and this action is transitive, we need only consider the faces of $\operatorname{conv} H_4$ containing $\mathbf{I}$. This means that we are in fact interested in whether the extreme rays of the cone

$$\{S \in \text{End } V : \forall g \in H_4 \ \ (S, g) \le (S, \mathbf{I})\}$$

are of rank 1. So we stay in dimension 16 (instead of going to dimension 17 while homogenizing). The memory requirements are reduced substantially in this way. Nevertheless, the computation still takes too long.

A very important tool in reducing the amount of calculations in cdd is the choice of ordering of the inequalities describing $G^\circ$, i.e., the choice of ordering in the Coxeter group $G$. It seems that a clever choice of ordering could produce a sharp drop in the amount of extraneous rays.

The cdd program calculates elements of $\operatorname{Extr}(H_4)^\circ$. But we already know a lot of these elements, since $\mathcal{B}_{H_4}$ is a subset of $\operatorname{Extr}(H_4)^\circ$. So what we actually need is not the calculation of all extreme vectors but rather a verification that the convex hull of known extreme vectors is already the set we are studying.

11.1. **Birkhoff faces.** Here we outline another approach to our problem. Let us say that a face of $\operatorname{conv} G$ is a **Birkhoff face** if some Birkhoff tensor is orthogonal to this face. Because $H_4$ is non-branching, the group generated by the operator $T \mapsto T^*$ and pre- and post-multiplications by elements of $H_4$ acts transitively on the set of Birkhoff tensors, so each Birkhoff face can be transformed to any other Birkhoff face by this group. So we may consider only Birkhoff faces containing $\mathbf{I}$. It is easy to list all operators from $G$ belonging to a Birkhoff face containing $\mathbf{I}$ orthogonal to $\omega \otimes \tau \in \mathcal{B}_G$ :

$$\langle g\omega, \tau \rangle = (g, \omega \otimes \tau) = (\mathbf{I}, \omega \otimes \tau) = \langle \omega, \tau \rangle = 1 \text{ if and only if}$$

$$g = hk, \;\; h \in \operatorname{Stab}_G \tau, \;\; k \in \operatorname{Stab}_G \omega.$$

To confirm the Conjecture we need to show that every Birkhoff face is adjacent to only Birkhoff faces, i.e., every subface of a Birkhoff face comes from the intersection with another Birkhoff face. So we need to study the adjacencies of Birkhoff faces. We have written a computer program computing the graph of adjacencies of Birkhoff faces. It still takes too long to run it for $H_4$. Description of Birkhoff subfaces (i.e., intersections of Birkhoff faces) is a challenging problem closely related to many interesting topics, including a Word Problem for Coxeter groups, Bruhat orderings, etc.

11.2. **An application of Poincaré's Theorem.** Let us describe another possible approach. Consider the following differential 15-form on the 16-dimensional space $\operatorname{End} \mathbf{R}^4$ :

$$\Omega(x) = \sum_{i=1}^{16} \frac{x_i}{\|x\|^2} dx_1 \wedge \cdots \wedge^i \cdots \wedge dx_{16}.$$

As usual, $\wedge^i$ means that $dx_i$ is omitted. This form is orthogonally invariant, closed outside of the origin and its integral over any 15-dimensional surface, which is starlike with respect to the origin, is non-negative. By the Poincaré Theorem the integrals of this form over the boundaries of all 16-dimensional bodies containing the origin are the same. In particular they are equal to the easily computable integral over a sphere centered at the origin. Let's suppose that we could calculate $\int_\Phi \Omega$ where $\Phi$ is a Birkhoff face. Since the form is orthogonally invariant the integral over the union of all the Birkhoff faces of $\operatorname{conv} H_4$ equals this integral multiplied by the number of Birkhoff tensors. Since the integral over any face of $\operatorname{conv} H_4$ is non-negative, we conclude that all faces are Birkhoff faces if and only if the integral over the union of Birkhoff faces equals the integral over a sphere.

This approach, while certainly interesting, is still not efficient enough. To perform the needed numerical calculations would require integrating over a Birkhoff face of $\operatorname{conv} H_4$. But to numerically integrate, we have to be able to test whether a point is in a face of $\operatorname{conv} H_4$. Knowing the vertices of a face isn't enough to perform this test; we need to know the faces of a face, and this is again a problem for cdd. But there are 480 points in the 15-dimensional Birkhoff face of $\operatorname{conv} H_4$. And sadly, 480 vertices is still too many points for cdd to handle.

## 12. Open Problems

**Problem 12.1.** *Does Conjecture 1.4 hold for $H_4$?*

Of course this problem could be solved by brute force, by simply using more powerful computers. But it would be much more interesting and useful to optimize the cdd program, to find clever orderings, etc.

It would be very interesting to develop a version of the cdd program taking symmetries into account

**Problem 12.2.** *Find a "classification-free" proof of Conjecture 1.4.*

This is definitely the heart of the matter. We believe that a promising approach is further study of Birkhoff faces and their subfaces. We have a reason to believe that the structure of Birkhoff subfaces of lower dimensions is simpler.

**Problem 12.3.** *Calculate* Extr $G^\circ$ *for irreducible branching Coxeter groups.*

We believe that real progress in this problem will depend upon progress in the previous problem.

**Problem 12.4.** *Calculate* Extr env $G$ *for irreducible branching Coxeter groups.*

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