Counterexample for Polynomial Approximation with an order of magnitude bound on the C^m norm

September 7, 2008

Purpose The purpose of the following remarks are to provide a counterexample to a conjecture involving polynomial interpolation with control over the C^m norm. Also I'll pose a possible reformulation of the conjecture which may be true.

Notation We'll fix a C^m-norm given by $||F||_{C^m(X)} = \sup_{x\in X} \max_{|\alpha|\leq m} |F^\alpha(x)|$, where $X \subset \mathbb{R}^n$ is a domain, and F is m-times differentiable on X. Though we are now fixing a C^m-norm, the following remarks will hold true for any reasonable C^m-norm. We denote the space of Polynomials over $\mathbb{R}^n$ of degree $\leq D$ by $P_D(\mathbb{R}^n)$. We also denote the n-dimensional cube with center a and sidelength $2l$ by $Q(a,l)$, dependence on n is to be assumed depending on the context. When we say 'Data' we mean a finite set E, along with a function $f : E \to \mathbb{R}$.

1 The Conjecture

Now we state the conjecture.

Conjecture 1.1. *Fix $m \geq 0$, and $n, k \geq 1$. Then there exists $D = D(m,n,k)$, $C = C(m,n,k)$, such that $\forall E \subset Q(0,1)$ with $\#(E) = k$, and $\forall f : E \to R$ there exists a polynomial $P \in P_D(\mathbb{R}^n)$ satisfying the following properties:*

1. *$P|_E = f$.*
2. *$||P||_{C^m(Q(0,1))} \leq C \inf\{||F||_{C^m(Q(0,1))} : F \in C^m(Q(0,1))$, and $F|_E = f\}$*

The counterexample relies on the following classical inequality.

Theorem 1.2 (Markov's Inequality)**.** *Let $P \in P_D(\mathbb{R})$ then*

$$\sup_{[-1,1]} |P'| \leq \frac{D^2}{2} \sup_{[-1,1]} |P|. \tag{1}$$

2 The Counterexample

Theorem 2.1. *Let $m \geq 0$, and $n \geq 1$ be given. Let $k = 2(m+1)$. Then* (1.1) *fails for these values of m,n, and k.*

Proof. It suffices to consider $m \geq 0$, and $n = 1$, since the counterexample can be trivially extended to arbitrary $n \geq 1$.

We will let C_i, and D stand for the controlled constants; that is constants that depend only on m, n, and k (Though this dependence will be omitted in what follows.)

We will take ϵ with $0 < \epsilon < \frac{1}{m+1}$ to be fixed later. We now define sets E_ϵ, and functions f_ϵ.

1. $E_\epsilon = \{-(m+1)\epsilon, -m\epsilon, ..., -\epsilon, \epsilon, m\epsilon, (m+1)\epsilon\}$.
2. $f_\epsilon(x) = -x^m$ if $x \in E_\epsilon \cap [-1, 0]$, and $f_\epsilon(x) = x^m$ if $x \in E_\epsilon \cap [0, 1]$.

Now assume that $\exists P_\epsilon \in P_D(\mathbb{R})$ satisfying the properties of (1.1) with data E_ϵ, and f_ϵ.

We note the following property of f_ϵ:

$$\inf\{||F||_{C^m([-1,1])} : F \in C^m([-1,1]), F|_{E_\epsilon} = f_\epsilon\} = C_0 \tag{2}$$

And therefore because P_ϵ satisfies the conditions of (1.1) we have that:

$$\begin{aligned} ||P_\epsilon||_{C^m([-1,1])} &\leq C \inf\{||F||_{C^m([-1,1])} : F \in C^m([-1,1]), F|_{E_\epsilon} = f_\epsilon\} \\ &= CC_0 = C_1 \end{aligned} \tag{3}$$

Now assume that $\exists P_\epsilon \in P_D(\mathbb{R})$ satisfying the properties of (1.1). Then we see by repeated application of the mean value theorem that $\exists x_0 \in [-(m+1)\epsilon, (m+1)\epsilon]$ such that:

$$|P_\epsilon^{(m+1)}(x_0)| > \frac{C_1(m)}{\epsilon} \tag{4}$$

Now (1.2), and (3) imply the following:

$$\begin{aligned} \sup_{[-1,1]} |P_\epsilon^{(m+1)}| &\leq \frac{D^2}{2} ||P_\epsilon^m||_{C([-1,1])} \\ &\leq \frac{D^2}{2} C_1 \\ &= C_2 \end{aligned} \tag{5}$$

Fixing $\epsilon < \frac{C_1}{C_2}$, we see that (4) implies the following:

$$\sup_{[-1,1]} |P_\epsilon^{(m+1)}| > \frac{C_1}{\epsilon} > C_2 \tag{6}$$

Thus we have a contradiction for any arbitrary $m \geq 0$, and $n = 1$. To extend this counterexample to arbitrary $n >= 1$ we can let our counterexample be supported only on the x_1-axis of $\mathbb{R}^n$, it is not too hard to check that all the important properties are unaffected by the presence of the additional $n - 1$ variables.

Thus we have our contradiction for arbitrary $m \geq 0$, $n \geq 1$, and $k = 2(m+1)$.

□

There is a way to modify the conjecture so that Markov's Inequality doesn't provide such an immediate roadblock. First a definition:

Definition 2.2. *Given $\delta > 0$. If $E \subset \mathbb{R}^n$, we say that E is δ-separated if $|x - y| > \delta$, $\forall x, y \in E$ with $x \neq y$.*

Now for the proposed conjecture:

Conjecture 2.3. *Fix $m \geq 0$, and $n, k \geq 1$. Also, fix $\delta > 0$. Then there exists $D = D(m, n, k, \delta)$, $C = C(m, n, k, \delta)$, such that given $E \subset Q(0, 1)$ which is δ-separated, satisfying $\#(E) = k$, and given $f : E \to \mathbb{R}$. Then there exists a polynomial $P \in P_D(\mathbb{R}^n)$ satisfying the following properties:*

1. $P|_E = f$.
2. $||P||_{C^m(Q(0,1))} \leq C \inf\{||F||_{C^m(Q(0,1))} : F \in C^m(Q(0, 1)),$ *and* $F|_E = f\}$